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ON CONVEX STRATIFICATIONS

Paper by Bruno De Finetti (Trieste)

Sumary. - Given a convex function, the sets where it assumes values greater than a generic constant c are obviously convex sets, each of them inside another; but the ~~contary~~ converse is not true, i. e. that ~~to~~ with such a class of sets we can always associate a convex function. We are studying the circumstances upon which the exceptions depend and the conditions which can exclude them.

I GENERAL REMARKS

Let f(P) be a convex function of the points P (of the plane, or, in general, of an affine space of any finite number of dimension.(1) Then the regions defined by the inequalities f(P) $\geq$ c form obviously, (as the constant c changes) a family of convex regions, one ~~internal to the~~ inside another, or, as we can say ~~shortly~~, ~~given~~ a convex stratification is it possible to associate with it, in the ~~way we saw before~~, a convex function f(P)? ~~That is~~, in brief, is such a stratification a stratification of convex function?

Generally we seem to think so: for instance in ~~the~~ mathematical e conomics ~~we think it possible to derive~~, from the fact that the regions φ(P) $\geq$ c (Precisely: the regions bounded by the "indifference varieties") are convex, that f(P) (the index of ~~economic~~ utility) can be a ssumed convex ~~too, (that is~~ either φ(P) is convex or we may replace it by f(P) = F (φ (P)), F being increasing, ~~so~~ that f(P) ~~might be~~ convex).

Geometrically, in the more intuitive case of the plane, such a statement would mean that, given as contours for a surface z = f(P) = f(x,y) a family of convex curves φ (P)= φ (x,y) = const., it is always possible [by making use of the resulting arbitrariness of f, which is defined ~~except for~~ an increasing transformation, f = F(φ)] to find a convex surface z = f(P) having the given contours. Such a property holds true when, for instance the function φ (x,y) is ~~supposed~~ to have bounded first and second deriviatives: that is clear if we consider ~~f = F(φ) = -e~~ which, in such a case,

(1) or also infinite: in such a case - we have to give up the conclusions resulting from being able to talk about "maxima" rather than "superior extreme" in several questions (see for instance note (7)).

$f = F(\varphi) = -e^{-\lambda\varphi}$

is certainly convex provided that λ is large enough.(2) Such a conclusion would not be valid if a restriction of this kind were not imposed. In Section 2 we shall consider some negative instances from which to start in order to face the problem in its general terms and to study the meaning of the circumstances which prevent the existence of the discussed convex function (or more precisely which cause it to degenerate into a constant.)

In any case, if the problem admits a solution, there is one among the solutions (univocally determined except for an additive and a multiplicative constant: z=h+kf(P)) which is the << least of all>> (in the intuitive sense which will be made precise in Section 3); every other solution is given by F(f). F being increasing and convex. Some questions - relating to these minimal convex functions - which may be interesting independently of the problem previously considered, will be studied in sections 5 and 7.

2. Examples

Let us consider some examples of convex stratifications which are not stratifications of convex functions. For the sake of simplicity we will deal only with circular plane stratifications (families of circles each internal to the other.).

Figures 1 and 3 represent solids which have been obtained from cones by changing their profiles as it appears in the section: the plane of the section is a plane of symmetry: therefore the contours are the circles whose diameters are indicated in section.

The alteration of the cone which is shown by the figure 1 has been obtained by substituting 1/4 of a circle to a part of the profile. In the figure 3 a part of the

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(2) If $\psi(x)$ is a function of one variable, the second derivative of $f(x) = -e^{-\lambda\psi}$ is $f''(x) = -\lambda e^{-\lambda\psi}(\lambda\psi'^2 - \psi'')$ which is always negative provided: $\psi(x)$ is twice differentiable, the ratio ψ''/ψ'^2 is bounded above in the field which is considered, and λ is superior to the upper extreme of such a ratio. We can argue in an analogous way for $\psi(x,y)$ as a function of two variables (or of three or more): the condition becomes:

$$\lambda \geq \sup\left\{\frac{\partial^2\psi}{\partial p^2}\Big/\left(\frac{\partial\psi}{\partial p}\right)^2\right\}$$

for every point of the field and every direction p (or, also for the only direction having the maximum slope, if the theorem of No. 4 is taken into account.)
We shall see in the note (9) that the only existence of second derivatives without the additional restriction that they be bounded, is no longer a sufficient condition for the established conclusion.

profile has been divided in an infinite number of parts (for instance, each being one half of the preceding) on which notches, ~~like the one represented by~~ figure 3a, have been made.

The solid represented by figure 2 can be also considered ~~derived~~ from a cone. For that purpose the cone has been atered - not in such a symmetric way - but by shifting the successive sections, so that the centers (in projection) move along a logarithmic spiral (radius of the circles = length of the spiral from the asymptotic point.)

The stratifications, which are obtained from the examples represented by the figures 1 and 2, have some necks: in the two points A and B of the figure 1(3) the strata become "infinitely subtle" (we ~~shall say the same thing~~ in a precise way in the sections 6 and 7);

The same fact can be seen in figure 2 on all the points of a curve (logarithmic spiral whose osculating circles are represented by the given circles of which the spiral is the envelope curve.)

This correspond, to points of the solid ~~which have an~~ infinite or zero slope ~~and~~ which cannot be eliminated by any ~~change of shape~~ $f=F(\varphi)$ (as it is obvious and we shall see anyway, in section 6) and ~~prevent~~ convexity. In the case ~~shown by~~ figure 3, in order to restore convexity by correcting the effect of one of the notches, we should quadruple (while ~~keeping still~~ the upper part of the graph) the height of all the part below the throat of the notch (as we can see in figure 3b). But if we repeat such a process ~~for~~ an infinite number of times, the frustrum of the cone with the notches is transformed into a solid whose ~~length~~ extends to infinity, and which therefore cannot be joined with the lower part.

3. DEFINITION; THE MINIMAL CONVEX FUNCTION

To deal with the problem we shall place ourselves in the more general affine space S: given in S points $P_1, \ldots P_n$ and numbers $\lambda_1, \ldots \lambda_h$ ($\Sigma_h \lambda_h = 1$) we can define their

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(3) A and B are the extremes of the part substitued by an arc of circle (in the figure the letters are not indicated).

linear combination (barycenter) $P = \sum_h \lambda_h P_h$; as we do not assume metrical notions, we may think it possible to compare the lengths of two segments only if they are parallel. A set C is defined as <u>convex</u>[4] if it includes all the segments whose extremes are contained by the set (i.e.: together with P_1 and P_2, every $P = \lambda P_1 + \mu P_2$ with $\lambda, \mu > 0$, $\lambda + \mu = 1$; then also: together with $P_1 \ldots P_n$, every $P = \sum_h \lambda_h P_h$, with $\lambda_h > 0$, $\sum_h \lambda_h = 1$). A real function f(P), defined on a convex set C, is said to be <u>convex</u> there if $f(P) \geq \sum_h \lambda_h f(P_h)$ - whichever are the assumed points $P_1, \ldots P_n$ of C - and $\lambda_1 \ldots \lambda_n$ being positive ($\sum_h \lambda_h = 1$); and P being equal to $\sum_h \lambda_h P_h$.

It is useful to note the lemma: given in C, any function $\psi(P)$ (not convex) and supposing:

"$f(P) = \sup. \sum_h \lambda_h \psi(P_h)$ as we vary all the possible ways of expressing $P = \sum_h \lambda_h P_h$ as a linear combination with coefficients $\lambda_h > 0$ of any finite number of points P_h of C",

it follows that f(P) is convex, and that, in all C, we get $f(P) \geq \psi(P)$, while $f(P) \geq \psi(P)$ when $\psi(P)$ is any other convex function $\geq \psi(P)$.

The demonstration is obvious.

Besides the level varieties f(P) = const. let us consider the <u>strata</u> $a \leq f(P) \leq b$ of a function f(P): set of the points P where f(P) assumes the values in a given interval (a,b); we shall call <u>stratification</u> the subdivision in strata obtained in such a way.[5] A stratification can be called <u>convex</u> if every stratum is the difference of two convex sets (in the examples of the section 2: the zone between two non secant circles). A (convex) function $\varphi(P)$ is defined to be <u>minimal convex</u> in one of its strata $a \geq (P) \leq b$ - (and therefore, obviously, in every stratum which is contained in the one considered) - if every convex function f(P) - having the same stratification [the same level variety, i.e. $f(P) = F(\varphi(P))$ F being increasing] and the same values on the level varieties [$f(P) = a$ for $\varphi(P) = a$, $f(P) = b$ for $\varphi(P) = b$ i.e. $F(a) = a$, $F(b) = b$] - is, in the stratum, $\geq \varphi(P)$.

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(4) For general notions, see for instance T. Bonnesen u. W. Fenchel, Theorie der Konvexen Korper, Berlin, Springer, 1934.

(5) If we want to give an abstract definition of that: the class D of the

We now give two different ways to draw the minimal convex function when any set of level varieties is given (the two bounding the stratum, with those in the inside or not, or a finite number or a numerable infinity,..., or all); and given the extreme values a, b of the contour ($a<b$).

The first way is an iterative process. Let us start from the function

$$\varphi_0(P)=\begin{cases} a \text{ in the entire stratum, except} \\ \text{on the inside contour where it is equal to } b. \end{cases}$$

Let $\varphi_1(P)$ be the minimal convex function $\geq \varphi_0(P)$ (see the preceeding lemma), and then let $\varphi_2(P)$ be the minimal function $\geq \varphi_1(P)$ which is constant on the prescribed level varieties (it is enough to take $\varphi_2(P) = \sup \varphi_1(Q)$, Q being on the level varieties passing through P or external to P).

Analogously let us pass from φ_2 to φ_3 and from φ_3 to φ_4, in general from φ_{2n} to φ_{2n+1} and from φ_{2n+1} to φ_{2n+2}; the sucession is never decreasing and therefore it approaches a limit function $\varphi(P)$, which is convex being the limit of the convex functions φ_h (h being odd), and constant on the established level varieties being the limit of the φ_h (h being even) possessing such a property.

In order that the solution should not be illusory, φ must not degenerate into a constant; the only way this can happen is that $\varphi(P) = b$ all along a stratum rather than only on the inside level variety, because, if $\varphi(Q)$ is equal to c for a point Q inside the set $\varphi(P) \geq c$, we cannot have in any point $\varphi(P) > c$ (to prove it, let us assume $\varphi(P) > c$; by lengthening the segment PQ beyond Q, which has been assumed inside the set where $\varphi \geq c$, we shall find more points R, $\varphi(R)$ being $\geq c$ and for Q ranging between R and P we should have $\varphi(Q) > c$ contrary to our hypothesis.)

Moreover, it is self evident that if we change the extreme values a and b to a' and b' ($b'>a'$ always) all the φ_h and the φ change linearly (because all the procedures are

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(5 continued)
difference sets of a class of sets K, each inside the other; i. e., K is such that for two of its sets A and B, whatever they may be, $A \supset B$ always or $B \supset A$ (or also: K results ordered with regard to $\supset$)

of ~~the same kind~~ an affine character): φ becomes $\varphi' = h + k\varphi$ ~~supposing~~ with $h = (ba' - ab')/(b - a)$ and $k = (b' - a')/(b - a)$ (so that $h + ka = a'$, $h + kb = b'$); the minimally convex function relative to a stratification is therefore, as ~~we said before~~ announced, determined ~~except for an~~ to within an additive and multiplicative constant (> 0).

4. CONVEXITY OF THE "PROFILES"

The other procedure - that is more analytic-constructive- to determine $\varphi(P)$ requires that we take into consideration the <u>thickness</u> of the strata and the <u>profile</u> of the function according to its different <u>giaciture</u>.(*)

Let $\xi(P)$ be a linear function of P: i.e. let $\xi(\Sigma_h \lambda_h P_h)$ be equal to $\Sigma_h \lambda_h \xi_h(P_h)$. The parallel hyperplanes ξ = const. define a <u>giracitura</u>; we shall consider as included in the notion also that of the orientation given by the direction along which ξ increases (therefore $\xi' = h + K$ will define the same giacitura of ξ for $k > 0$, and for $k < 0$ the opposite giacitura: the same hyperplanes, the orietation inverted). We shall call "ξ - thickness of "a stratum" the difference $\xi'' - \xi'$ where ξ'' and ξ are the upper bounds of $\xi(P)$ on the pypersurfaces bounding the stratum on the outside and inside respectively. The ratio of the ξ thicknesses of several strata no longer depends on the actual coefficient $k > 0$ but only on the (orientated) giacitura. In this sense we could speak of the ratio between the thicknesses of several strata according to a given giacitura (and, sometimes, for the sake of simplicity, we shall speak of "thicknesses" implying that, they being determined except for a constant, we should limit ourself to consider ratios between thicknesses taken according to the same giacitura).

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(*) "<u>Giacitura</u>" (pl giaciture) means literally "situation". Since the English equivalent does not seem appropriate here, and since moreover the mathematical definition is given in the following line of the text (v. supra) the original Italian has been kept throughout. (translator's note.)

Now let $f(P)$ be a function of P, then we shall indicate with $f_\xi(x)$ the upper extreme of $f(P)$ on the hyperplane $\xi(P) = x$; the function $f_\xi(x)$ is the <u>profile</u> of the function f according to the giacitura ξ (this denomination is of a intuitive meaning in the case of a surface $z=f(P)$, P being points on the plane x,y).

Let us prove now, as a <u>lemma</u>, that if (for a function $f(P)$, its level varieties ($f(P) = c$) bound convex regions ($f(P) \geq c$), the necessary and sufficient condition that $f(P)$ be a convex function, is that all its profiles be convex. In fact, if $f(P)$ is assumed to be convex, it follows that also $f_\xi(x)$ is convex, that is

$$f_\xi(x_2) \geq \lambda f_\xi(x_1) + \mu f_\xi(x_3) \quad (\text{for } x_1 < x_2 < x_3,\ x_2 = \lambda x_1 + \mu x_3,\ \lambda + \mu = 1),$$

because, having indicated by P_h (for h ranging from 1 to 3) the point of $\xi = x_h$ where $f(P)$ assumes the maximum value $f_\xi(X_h)$, and having supposed $Q = \lambda P_1 + \mu P_3$, it follows, from (the linearity of ξ) that $\xi(Q) = X_2$ so that $f_\xi(x_2) \geq f(Q)$ while, from the convexity of f, $f(Q) \geq \lambda f(P_1) + \mu f(P_3)$, that proves our assertion.

Conversely if we assume that $f(P)$ is <u>not</u> convex and that every region bounded by level variety is convex, it follows that at least one profile is not convex. In fact, let us, for instance, suppose that there are three points, P_1, P_3 and $P_2 = \lambda P_1 + \mu P_3$ ($\lambda, \mu > 0$; $\lambda + \mu = 1$) such that $f(P_2) < f(P_1) + f(P_3)$ and let $\xi = x_2$ be a supporting hyperplane of the level variety at P_2:(6) then $\xi(Q) \leq x_2$ for every Q where $f(Q) \geq f(P_2)$, and therefore $f_\xi(x_2) = f(P_2)$. Since for $x_1 = \xi(P_1)$ we have $f_\xi(x_1) = f(P_1)$ and the same for $x_3 = \xi(P_3)$ the profile according to the giacitura ξ is not convex because:

$$f_\xi(x_2) < \lambda f_\xi(x_1) + \mu f_\xi(x_3),\ x_2 = \lambda x_1 + \mu x_3.$$

We must remark that a convex function $f(P)$ remains univocally determined if we know all its profiles $f\ (x)$: : in fact $f(P)$ is equal to min. $f_\xi(\xi(P))$ as ξ changes, or also $f(P) = f_\xi(x)$ if $\xi = x$ is supporting hyperplane at P (for in such a case we have the minimum).

5. CONDITIONS FOR THE PROFILES

the condition of convexity of the profiles may be written:

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(6) That is the tangent hyperplane if the point is not angular; in the other cases anyone of the planes which touch the variety in that point without intersecting it; i.e. a supporting hyperplane.

$$\frac{f\ (x_2) - f\ (x_3)}{x_3 - x_2} \geq \frac{f\ (x_1) - f\ (x_2)}{x_2 - x_1} \qquad (x_1 < x_2 < x_3)$$

or in brief:

$$\frac{\delta_2}{s_2} \geq \frac{\delta_1}{s_1} \qquad \text{or} \qquad \delta_2 \geq \delta_1 \frac{s_2}{s_1}$$

s_1 and s_2 being the thicknesses, according to the giacitura , of the two strata, and δ_1 and δ_2 being the corresponding increments of the function f. Since this holds for all giaciture, if f is convex, then we must have:

$$\delta_2 \geq \delta_1 \quad \max \frac{\sigma_2}{\sigma_1}$$

σ_2 and σ_1 being the thicknesses of the two strata according to a generic giacitura and the maximum having been taken as the giacitura charges.[7] By considering successive strata we can state inductively:

$$\delta_n \geq \delta_1 \max\left(\frac{\sigma_2}{\sigma_1}\right) \cdot \max\left(\frac{\sigma_3}{\sigma_2}\right) \cdot \ldots \cdot \max\left(\frac{\sigma_n}{\sigma_{n-1}}\right),$$

or, in a different way:

$$\frac{\delta_n}{s_n} \geq \frac{\delta_1 s_1}{s_1 s_n} \max \frac{(\sigma_2)}{(\sigma_1)} \ldots \max \frac{(\sigma_n)}{(\sigma_{n-1})}$$

or also

$$\frac{\delta_n}{s_n} \geq \frac{\delta_1}{s_1} \left\{ \frac{\max(\sigma_2/\sigma_1)}{s_2/s_1} \cdot \frac{\max(\sigma_3/\sigma_2)}{s_3/s_2} \cdot \ldots \cdot \frac{\max(\sigma_n/\sigma_{n-1})}{s_n/s_{n-1}} \right\}.$$

Since f_ξ, being convex, is differentiable (except, at the most, for a denumerable infinity of angular points which we shall later exclude from acting as subdivisions for the strata) and the derivative is increasing, we have:

$$(4) \qquad -f_\xi'(x_{n-1}) \geq \frac{\delta_n}{s_n} \geq \frac{\delta_1}{s_1} \left\{ \prod_{1/h}^{n-1} \frac{\max(\sigma_{h+1}/\sigma_h)}{s_{h+1}/s_h} \right\} \geq -f_\xi'(x_1) \left\{ \prod_{1\ h}^{n-1} \frac{\max(\sigma_{h+1}/\sigma_h)}{s_{h+1}/s_h} \right\},$$

that is: the ratio between the derivative of f (x) at two points whatever x_2 and x_1 ($x_2 > x_1$) is $\geq$ than the product $\{\prod \ldots\}$ corresponding to any subdivision in strata of the considered stratum and therefore it is also $\geq$ than $W_\xi(x_1, x_2) = \sup. \{\prod \ldots\}$ as the

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[7] The existence of a maximum can be proved since we are dealing with convex functions.

subdivision is convex and therefore it is so, in particular, if $f_\xi(x)$ is the solution of the differential equation:

$$\psi'_\xi(x) = k\, W_\xi(x_1, x) \qquad (k = \psi'_\xi(x_1))$$

that is for:

$$(5) \qquad f_\xi(x) = \psi_\xi(x) = h + k \int_{x_1}^{x} W_\xi(x_1, u)\, du.$$

It is enough to point out that the (5) is independent of the giacitura, that we can derive (4) from it directly for s_h corresponding to any giacitura, and that $\{\prod \ldots\}$ is in it ≥ 1 (product of factors all ≥ 1): therefore the ratios of the increments, and hence the derivatives are increasing.

Moreover we shall prove that $\psi(P)$, determined by the profiles $\psi_\xi(x)$, is minimal convex (and therefore coincides, necessarily, with the solution which has been determined in another way in section 3). It is enough to prove that for $f(P)$ convex (and having the contour values a and b in common with $\psi(P)$, we cannot have in any point Q, $f(Q) < \psi(Q)$; were it so and should we suppose $f = F(\psi)$, $\psi(Q) = q$, then $F(q)$ would be $< q$ while $F(a) = a$, $F(b) = b$, and hence there would exist p and r, $a<p<q<r<b$, such that $F'(p) < F'(r)$.

But in such a case $f_\xi(x) = F(\psi_\xi(x))$, $f'_\xi(x) = F'(\psi_\xi(x))\, \psi'_\xi(x)$, and the ratio between the derivatives in two points for which $\psi_\xi(x_2) = p$, $\psi_\xi(x_1) = r$ would be:

$$\frac{f'_\xi(x_2)}{f'_\xi(x_1)} = W_\xi(x_1, x_2)\, \frac{F'(p)}{F'(r)} < W_\xi(x_1, x_2),$$

therefore f_ξ cannot satisfy the prescribed condition.

It is well, to note, as a corollary, the following property which characterizes the minimal convex functions: ψ is minimal convex if and only if, for all F increasing and convex, $f = F(\psi)$ is convex.

It is often useful to replace the consideration of $W_\xi(x_1, x_2)$ which depends on the level varieties V_1 and V_2, on which $\max \xi = x_1, x_2$, and also on the giacitura ξ, by the consideration of $W(V_1, V_2) = \max W_\xi(x_1, x_2)$ given by:

$$(6) \qquad W(V_1, V_2) = \sup. \prod_{1}^{n}{}_{h} \max\left(\frac{\sigma_{h+1}}{\sigma_h}\right)$$

with the conventions $\sigma_{n+1} = \sigma_1$; the remarkable intrinsic significance of $W(V_1,V_2)$ is the following: the ratio $f_\xi'(x_2)/f_\xi'(x_1)$ has a maximum $\geq W$ (= if and only if f is minimal convex).

6. EXISTENCE OF CONVEX FUNCTIONS FOR A GIVEN STRATIFICATION

The fact which we are studying, that is the existence of a nondegenerate convex function with the given level varieties, appears now to be obviously bound to the circumstance that W be limited. If $W < K$, there is no danger of degeneration. If $W_\xi(x_1,x_2) = \infty$, it follows $\psi'_\xi(x_2)/\psi'_\xi(x_1) = \infty$ and therefore either $\psi'_\xi(x_2) = \infty$ or $\psi'_\xi(x_1) = 0$ (or both); in such conditions a convex function answering the problem in the stratum (V_1, V_2) may still exist (as in the example of figure -1- which we shall call briefly example 1, for the stratum having a rounded profile), but such a function cannot be found in a larger stratum.

In order to have $W_\xi(x_1, x_2) = \infty$, since $W_\xi(x_1, x_2) = W_\xi(x_1, x) \cdot W(x,x_2)$ for any $x_1 < x < x_2$, W_ξ must be infinite for one at least of the two subintervals; if we proceed in such a way it follows that there must be in (x_1, x_2) a x such that $W_\xi(x', x'') = \infty$ (and therefore $W(V',V'') = \infty$) for every interval (x',x'') containing x. The corresponding level variety will be called <u>exceptional</u>; we can thus state that in order to admit a convex function it is necessary that a convex stratification does not contain any exceptional variety inside the considered stratum (and it is sufficient that in addition the contour varieties should not be exceptional.)

In order to comprehend and to classify the "exceptionalities" so defined, it is useful to consider, in general, for every level variety V_0, the lower bound of $W(V',V'')$, (V', V'' including V_0); let us denote it by "$W(V_0)$". We must always have $W \geq 1$ (W being obviously ≥ 1); beside the case of the exceptional varieties, already considered, where $W = \infty$, we want to distinguish the cases of the <u>regular</u> varieties where $W = 1$, and of the <u>corner</u> varieties, where $W > 1$ but finite. The ground for such denominations is self evident: we can apply the appellation "corner" to those varieties of which at least one profile has angular point (W being the ratio between the slope of the tangents on the right and on the left, according to the profile which causes this ratio to be a maximum). Analogously, for the exceptional varieties, we have at least one point where such a ratio becomes infinite, and the slope is necessarily zero on the left if it is finite on the right, and infinite on the right if it is finite on the left. We shall distinguish two cases of exceptionalities: we shall have neck variety, when, by the thicknesses of two strata, we can already make the product

$$(7) \qquad \max\left(\frac{\sigma_2}{\sigma_1}\right) \cdot \max\left(\frac{\sigma_1}{\sigma_2}\right) = \frac{\max(\sigma_2/\sigma_1)}{\min(\sigma_2/\sigma_1)}$$

as large as we want.

Such a circumstance occurred in the examples 1 and 2 (sect: 2): inside a stratum taken as subtle as we want and including a neck we can always find two strata whose thicknesses are such that their ratio, according to a giacitura, becomes as small as we want, compared to the giacitura for which it is a maximum.

In the contrary case we shall have <u>instability varieties</u> (8), which can be varieties formed as accumulations of corner varieties (as in example 3 of section 2), but they can also be inside a stratum where all other varieties are regular (it would be enough in example 3 to make the vertices of the neck round) (9). An accumulation variety of corner edge varieties is necessarily exceptional if the product of their W diverges, in every neighborhood of it (see example 3), otherwise it can be also regular (we would only need, always in the example 3, to make the notches more and more smooth),

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(8) This denomination has been conceived in order to notice how the thickness of a subtle stratum fluctuates irregularly while we let it approach the same variety. (see, for instance, figure 3).

(9) Let us remark that the profile having notches (always in the example represented by figure 3) can be altered in such a way as to have the second derivative exist and be finite everywhere (though it is not bounded) and to let the exceptionality hold.
It is enough for instance to make the notches more and more smooth but not too quickly, and precisely in such a way that the ratio between the slopes of two successive parts approaches one, but the infinite product of these ratios diverges, (for instance the ratio of the n.th notch = $1 + 1/n$), while the width of the notches decreases in an appropriate way (for the same example, the width of the n^{th} notch for instance of the order of $n^{-3/2}$), so that the order of magnitude of the distance from the point of accumulation of the notches (remainder of the sequence) is superior to that of the oscillations of the slope in consequence of the notches (in the instance, $n^{-1/2}$ in

or having corner edge (it would be enough, besides, to superpose an angularity).

7. PARTICULARITY OF BEHAVIOUR IN POINTS

While considering the examples of section 2, we have spoken about <u>notch points</u> (and not about notch varieties); and in general all the considered possible ways of behaviour are referred more specifically to points of the varieties where they appear. First of all let us acknowledge that on every level variety there are some points which we shall call <u>hinge points</u>, definable, with regard to the minimal convex function ψ, by any one of the following conditions which are equivalent:

- if a transformation $f = F(\psi)$ causes the function f not to be convex, there are some points P where f(P) is less than we would need to comply with the conditions for convexity: we shall call hinge points those for which this property holds necessary as soon as it can be proved true for one only of the points of the same level variety;
- by suppressing the constraint of respecting the stratification for the points of a field C, the minimal convex function ψ resulting may or may not be improvable: precisely, it is improvable if all the hinge points P are inside C at least for one level variety;
- the profile corresponding to the tangent giacitura (or to one, at least of the supporting giaciture) at the point P, has there a vanishing curvature;

(9 continued)
comparison with n-1). After that we can connect two successive sides of the broken line by curve arches (for instance by parabolic arches of the third degree) excluding the angular points; we obtain a curve having everywhere continuous first and second derivatives, except for the point of accumulation of the notches where the second derivative exists (and is zero) but neither continuous (or bounded) in any neighborhood. That proves, as we stated in the note (2), that we cannot suppress the condition of the second derivative being bounded in enunciating that sufficient condition.
It is not a necessary condition, of course; it is not even necessary that the first or second derivation should exist. Yet we can remark that differentiability alone is enough to exclude the possibility of corner varieties (except, eventually, on level varieties formed by <<stationary points>>, i. e. with a vanishing first derivation.)

- for such a giacitura the lower bound of

$$\frac{W_\xi(x_1, x_2) - 1}{x_2 - x_1}$$

decreases to zero as the neighborhood $x \pm \xi$ decreases to x (x_1, x_2 can be taken around $x = \xi$ (P)).

The last two formulizations are clearly equivalent, except for the geometric and analytic expressions; from the geometric formulization it clearly follows that, by making a profile concave in a point, the same must occur, on the level variety at that point, for all the profiles having there a vanishing curvature.

Such a consideration leads us to prove the affirmed existence of hinge points on every level variety; we can have at least one giacitura for which

$$\liminf \frac{W_\xi(x_1, x_2) - 1}{x_2 - x_1} = 0$$

(for x_1, x_2, appoaching, respectively from the left and from the right, the x of the considered level variety V_o).

In the contrary case we would have a number $\gamma > 0$ and a stratum containing V_o such that, for any giacitura ξ, x_1 and x_2 being taken in such a stratum, the expression of which we are considering the min. lim should result always $> \gamma$. It would then be:

$$\frac{\psi_\xi'(x_2)}{\psi_\xi'(x_1)} = W_\xi(x_1, x_2) > 1 + \gamma(x_2 - x_1)$$

and if we take $f = F(\psi) = \psi + 1/2\, K \psi^2$ (as constant being/> 0; F <u>not convex!</u>)

$$\frac{f_\xi'(x_2)}{f_\xi'(x_1)} = \frac{\psi_\xi'(x_2)}{\psi_\xi'(x_1)} \cdot \frac{1 + K\psi_\xi(x_2)}{1 + K\psi_\xi(x_1)} > [1 + \gamma \cdot \Delta x] \cdot 1 - \frac{\chi \Delta \psi_\xi}{1 + \chi \psi_\xi(x_1)} > 1 + \gamma \Delta K - \frac{K \Delta \psi_\xi}{1 + K \psi_1}$$

and therefore $f_\xi'(x_2) / f_\xi'(x_1) > 1$ if χ is small enough. Therefore ψ is not minimal convex (contradiction to the corollary of the section 6). Analogously on the neck varieties there are neck points where $W_\xi = \infty$, and on the corner varieties (or on neck or instability varieties) there are corner points where $W_\xi \neq 1$ (by defining, obviously W_ξ in an analogous way as W, but from W_ξ rather than from W).