Optimal Dividends: Analysis with Brownian Motion

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Abstract

In the absence of dividends, the surplus of a company is modeled by a Wiener process (or Brownian motion) with positive drift. Now dividends are paid according to a barrier strategy: Whenever the (modified) surplus attains the level b, the "overflow" is paid as dividends to shareholders. An explicit expression for the moment-generating function of the time of ruin is given. Let D denote the sum of the discounted dividends until ruin. Explicit expressions for the expectation and the moment-generating function of D are given; furthermore, the limiting distribution of D is determined when the variance parameter of the surplus process tends toward infinity. It is shown that the sum of the (undiscounted) dividends until ruin is a compound geometric random variable with exponentially distributed summands.

The optimal level b^* is the value of b for which the expectation of D is maximal. It is shown that b^* is an increasing function of the variance parameter; as the variance parameter tends toward infinity, b^* tends toward the ratio of the drift parameter and the valuation force of interest, which can be interpreted as the present value of a perpetuity. The leverage ratio is the expectation of D divided by the initial surplus invested; it is observed that this leverage ratio is a decreasing function of the initial surplus. For $b = b^*$, the expectation of D, considered as a function of the initial surplus, has the properties of a risk-averse utility function, as long as the initial surplus is less than b^*. The expected utility of D is calculated for quadratic and exponential utility functions. In the appendix, the original discrete model of De Finetti (1957) is explained and a probabilistic identity is derived.

1. Introduction

Recently, there has been much discussion in the United States about eliminating or reducing taxes on dividends so as to encourage corporations to pay more dividends. It may be a surprise to many that studies of optimal dividend payment strategies have appeared in the actuarial literature for half a century. A main purpose of this paper is to provide economic analyses of such strategies.

Traditionally, actuaries have been primarily concerned with the financial management of insurance companies and other financial systems, in particular with their solvency. In the classical model for determining the *probability of ruin,* the surplus of a company can increase without bounds. This is unrealistic. De Finetti (1957) suggested that other, more economic considerations such as dividend payments should also play an important role. Specifically, he considered a discrete-time model, in which the periodic gains of a company are $+1$ (with probability $\pi > \frac{1}{2}$) or -1 (with probability $1 - \pi$).

If the ultimate goal is to maximize the expectation of the discounted dividends paid to the shareholders of the company, what is the optimal dividend-payment strategy? De Finetti found that the optimal strategy must be a *barrier strategy,* and he showed how the optimal level of the barrier can be

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determined. De Finetti's idea inspired the pioneering work of Miyasawa (1962), Takeuchi (1962), and Morill (1966).

The problem of finding the optimal dividend-payment strategy has been discussed extensively by Karl Borch; see Borch (1974, 1990). The reader may also want to consult the monographs by Bühlmann (1970, sec. 6.4), Gerber (1979, sec. 10.1 and 10.2) and Seal (1969, pp. 163–6), and their references. Some recent papers on dividend-payment strategies are Asmussen and Taksar (1997), Paulsen and Gjessing (1997), Gerber and Shiu (1998, 2003a,b), Højgaard and Taksar (1999), Jeanblanc-Picqué and Shiryaev (1995), Siegl and Tichy (1999), Albrecher and Kainhofer (2002), Bühlmann (2002), Højgaard (2002), Claramunt, Mármol, and Alegre (2003), Irbäck (2003), and Lin, Willmot, and Drekic (2003). The reader should be cautioned that, in more general models, the optimal strategy can be a *band strategy* and not a barrier strategy.

In this paper, we go back to the roots and consider the continuous counterpart of De Finetti's (1957) model. Here it is assumed that the surplus of a company is a Wiener process (Brownian motion) with a positive drift. This model has the advantage that some very explicit calculations can be made. Furthermore, economic analyses of the results can be carried out more easily and in considerable depth. Basic mathematical results can be found in Gerber (1972), where the surplus process is the sum of a Wiener process and an independent compound Poisson process; but the paper provides no economic analyses.

Many results in this paper have their counterpart in the classical surplus model, where the aggregate claims are modeled as a compound Poisson process, in particular, when the individual claims are exponentially distributed. Since a Wiener process model can be obtained as a limit, such results in the classical model can be used to give an alternative, rigorous derivation for the corresponding results in the Wiener process model. As many actuaries are more familiar with the compound Poisson model, we shall point out such correspondences.

2. THE WIENER PROCESS MODEL AND BASIC RESULTS

Consider a company with initial surplus or equity $x > 0$. If no dividends were paid, the surplus at time t would be

$$X(t) = x + \mu t + \sigma W(t), \qquad t \geq 0, \tag{2.1}$$

with $\mu > 0$, $\sigma > 0$, and $\{W(t)\}$ being a standard Wiener process. This model can be found in Iglehart (1969), Grandell (1991), and Klugman, Panjer, and Willmot (1998). The company will pay dividends to its shareholders according to a barrier strategy with parameter $b > 0$. Whenever the (modified) surplus reaches the level b, the "overflow" will be paid as dividends. A formal definition can be given in terms of the running maximum

$$M(t) = \max_{0 \leq \tau \leq t} X(\tau). \tag{2.2}$$

Then the aggregate dividends paid by time t are

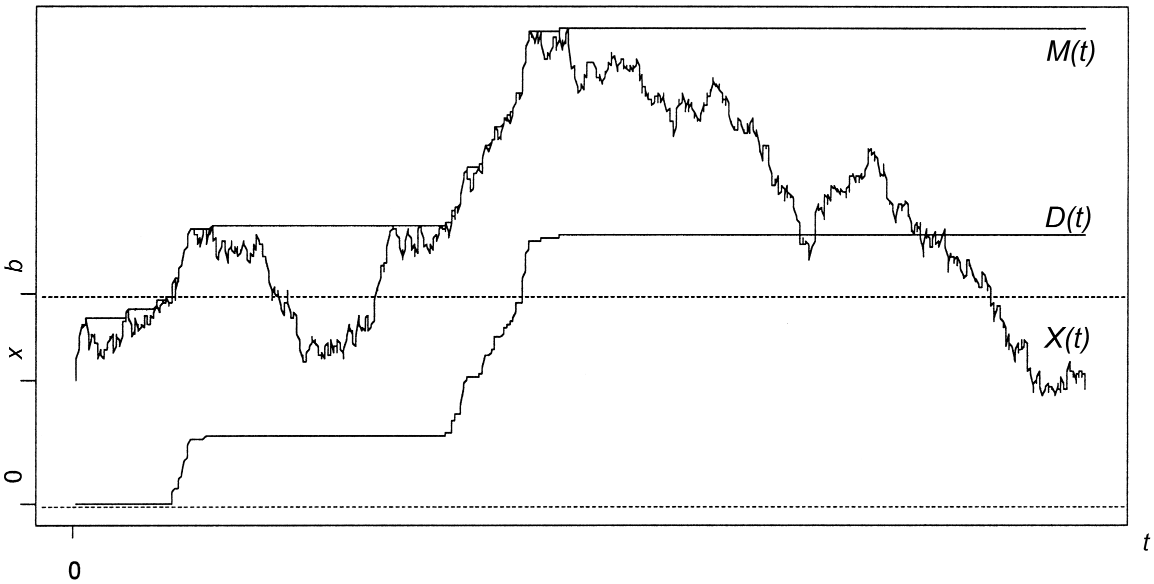
$$D(t) = (M(t) - b)_+ = \begin{cases} 0 & \text{if } M(t) \leq b \\ M(t) - b & \text{if } M(t) > b \end{cases} \tag{2.3}$$

(see Figure 1). It is assumed that the payment of dividends has no influence on the business; thus, the modified surplus at time t is $X(t) - D(t)$.

Let $\delta > 0$ be the force of interest for valuation, and let D denote the present value of all dividends until ruin,

$$D = \int_0^T e^{-\delta t}\, dD(t), \tag{2.4}$$

Figure 1
Illustration of Formulas (2.2) and (2.3)



where

$$T = \min\{t \geq 0 \,|\, X(t) - D(t) = 0\} \tag{2.5}$$

is the *time of ruin.* We use the symbol $V(x;\, b)$, $0 \leq x \leq b$, for the expectation of D,

$$V(x;\, b) = \mathrm{E}[D]. \tag{2.6}$$

As a function of the initial surplus x, $V(x;\, b)$ satisfies the homogeneous second-order differential equation

$$\frac{\sigma^2}{2} V''(x;\, b) + \mu V'(x;\, b) - \delta V(x;\, b) = 0, \qquad 0 < x < b. \tag{2.7}$$

This can be seen from the following heuristic argument. Let $0 < x < b$. In the infinitesimal time interval from 0 to dt, the surplus, with $X(0) = x$, does not reach either barrier (0 or b). Hence,

$$\mathrm{E}[V(X(dt);\, b)] = e^{\delta dt} V(x;\, b). \tag{2.8}$$

The right-hand side of (2.8) is

$$(1 + \delta dt)V(x;\, b) = V(x;\, b) + \delta V(x;\, b)dt.$$

Since

$$X(dt) = x + \mu dt + \sigma W(dt),$$

the left-hand side of (2.8) is

$$V(x;\, b) + \mu V'(x;\, b)dt + \frac{\sigma^2}{2} V''(x;\, b)dt.$$

Thus, subtracting $V(x;\, b)$ from both sides of (2.8) and then canceling dt yields (2.7).

The function $V(x; b)$ satisfies the boundary conditions

$$V(0; b) = 0, \tag{2.9}$$

$$V'(b; b) = 1. \tag{2.10}$$

Condition (2.9) is obvious: If $X(0) = x = 0$, ruin is immediate, and no dividends are paid. Condition (2.10) is a limiting case of formula (7.4) in Gerber and Shiu (1998). It can be explained as follows: Consider two situations, one with initial surplus $x = b$, and the other with initial surplus $x = b - \varepsilon$ ($\varepsilon > 0$ and "small"). Then, in the first situation, the dividends will be by the amount ε higher than in the second case, for almost all sample paths of $\{W(t)\}$. A rigorous proof of (2.10) can be found in Gerber (1972).

Subject to the boundary conditions (2.9) and (2.10), the solution of the differential equation (2.7) is

$$V(x; b) = \frac{g(x)}{g'(b)}, \qquad 0 \le x \le b, \tag{2.11}$$

where

$$g(x) = e^{rx} - e^{sx}, \tag{2.12}$$

with r and s being the roots of the quadratic equation

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - \delta = 0. \tag{2.13}$$

We let r denote the positive root and s the negative root,

$$r = \frac{-\mu + \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}, \tag{2.14}$$

$$s = \frac{-\mu - \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}, \tag{2.15}$$

so that both the numerator and the denominator in (2.11) are positive. Formula (2.11) can be viewed as a limiting case of Gerber and Shiu (1998, eq. 7.5).

We can rewrite (2.11) as

$$V(x; b) = \frac{g(x)}{g(b)}\frac{g(b)}{g'(b)} = \frac{g(x)}{g(b)}V(b; b), \qquad 0 \le x \le b. \tag{2.16}$$

Then we see that the ratio,

$$\frac{g(x)}{g(b)} = \frac{e^{rx} - e^{sx}}{e^{rb} - e^{sb}}, \qquad 0 \le x \le b, \tag{2.17}$$

can be interpreted as the expected discounted value of a contingent payment of 1, payable as soon as the surplus reaches level b, provided ruin has not yet occurred. Formula (2.17) is equivalent to (10.13.15) in Panjer (1998) and can be viewed as a limiting case of formula (6.25) in Gerber and Shiu (1998).

Remarks

In the remainder of this section, we consider the limiting case $\delta = 0$. Then $D = D(T)$, the total dividends paid until ruin, and $V(x; b) = \mathrm{E}[D(T)]$. From (2.14) and (2.15), we get $r = 0$ and $s = -2\mu/\sigma^2$. Hence,

$$g(x) = 1 - e^{-2\mu x/\sigma^2} \tag{2.18}$$

and

$$V(x;\, b) = \frac{\sigma^2}{2\mu}\,(e^{2\mu b/\sigma^2} - e^{2\mu(b-x)/\sigma^2}), \qquad 0 \le x \le b, \tag{2.19}$$

by (2.11). In particular,

$$V(b;\, b) = \frac{\sigma^2}{2\mu}\,(e^{2\mu b/\sigma^2} - 1), \qquad 0 \le x \le b. \tag{2.20}$$

It follows from (2.19) and (2.20) that

$$V(b;\, b) = V(b - x;\, b - x) + V(x;\, b), \qquad 0 \le x \le b. \tag{2.21}$$

This formula can be interpreted as follows. For $X(0) = b$, the total dividends paid until ruin can be decomposed as the sum of the total dividends paid until the modified surplus drops to the level x for the first time and the total dividends paid thereafter until ruin. Taking expectations, we obtain (2.21). Some actuaries may want to view (2.21) as the compound-interest formula

$$\bar{s}_{\overline{b}|} = \bar{s}_{\overline{b-x}|} + \bar{s}_{\overline{x}|}(1 + i)^{b-x},$$

where all interest functions are evaluated at the force of interest of $2\mu/\sigma^2$.

In the limit $\mu \to 0$ (in addition to $\delta = 0$), formula (2.19) becomes

$$V(x;\, b) = \mathrm{E}[D(T)] = x, \tag{2.22}$$

which is independent of b and σ. Then (2.21) is $b = (b - x) + x$.

From (2.17) and (2.18), we see that

$$\frac{g(x)}{g(b)} = \frac{1 - e^{-2\mu x/\sigma^2}}{1 - e^{-2\mu b/\sigma^2}}, \qquad 0 \le x \le b. \tag{2.23}$$

This is the probability that the Wiener process $\{X(t)\}$, with $X(0) = x$, $0 \le x \le b$, will reach level b before level 0. Formula (2.23) is well known; for example, it can be found in Karlin and Taylor (1975, chap. 7, theorem 5.2) and Harrison (1985, p. 43), and it is equivalent to (10.13.20) in Panjer (1998). The discrete counterpart of (2.23) can be found in some textbooks in the context of the *gambler's ruin* problem. In the terminology of risk theory, the factor $2\mu/\sigma^2$ is the *adjustment coefficient* and the function $e^{-2\mu x/\sigma^2}$ is the probability-of-ruin function $\psi(x)$; see also Klugman, Panjer, and Willmot (1998, corollary 6.10).

In the limiting case $\delta = 0$ and $\mu = 0$, (2.23) is

$$\frac{g(x)}{g(b)} = \frac{x}{b}, \qquad 0 \le x \le b. \tag{2.24}$$

3. THE DISTRIBUTION OF T UNDER A BARRIER STRATEGY

Consider that the barrier strategy with level b is applied. Thus, ruin is certain. We are interested in the distribution of the time of ruin, T. In this section, we calculate

$$L(x;\, b) = \mathrm{E}[e^{-\delta T}], \tag{3.1}$$

where $x = X(0)$ is the initial surplus or capital. This is the expected present value of a payment of 1 at the time of ruin and, at the same time, the *Laplace transform* of the probability density function of T. We shall also determine the expected time to ruin, $\mathrm{E}[T]$.

The functions $L(x;\, b)$ and $V(x;\, b)$ and the one defined by expression (2.17) are special cases of a family of functions $\{K(x;\, b)\}$, where

$$K(x;\, b) = \mathrm{E}[e^{-\delta\tau}K(X(\tau);\, b)], \qquad 0 \le x \le b, \tag{3.2}$$

with

$$\tau = \min\{t \ge 0 \,|\, X(t) = 0 \text{ or } X(t) = b\} \tag{3.3}$$

being the first time the surplus attains the level b or falls to 0. The argument we used to derive (2.7) also shows that $K(x;\, b)$ satisfies the homogeneous second-order differential equation

$$\frac{\sigma^2}{2} K''(x;\, b) + \mu K'(x;\, b) - \delta K(x;\, b) = 0, \qquad 0 < x < b. \tag{3.4}$$

Hence, the function $K(x;\, b)$ is a linear combination of the exponential functions e^{rx} and e^{sx}, with r and s given by (2.14) and (2.15), respectively. The coefficients of this linear combination depend on the boundary conditions. For $L(x;\, b)$, the boundary conditions are

$$L(0;\, b) = 1, \tag{3.5}$$

$$L'(b;\, b) = 0. \tag{3.6}$$

It follows that

$$L(x;\, b) = \frac{re^{-s(b-x)} - se^{-r(b-x)}}{re^{-sb} - se^{-rb}}, \qquad 0 \le x \le b. \tag{3.7}$$

Formula (3.7) can be found in Cox and Miller (1965, p. 233, ex. 5.6), which also indicates how the probability density function of T can be obtained by inverting (3.7). Note that $[1 - L(x;\, b)]/\delta$ is the expected present value of continuous payments at a rate of 1 from time 0 to T. In the limit $\delta \to 0$, we obtain

$$\mathrm{E}[T] = \frac{\sigma^2}{2\mu^2}\left[e^{2\mu b/\sigma^2} - e^{2\mu(b-x)/\sigma^2} - \frac{2\mu x}{\sigma^2}\right], \qquad 0 \le x \le b, \tag{3.8}$$

which matches equation (135) of Cox and Miller (1965, p. 235).

Remarks

(i) The compound Poisson counterpart of (3.7) has been given by Lin, Willmot, and Drekic (2003, eq. 5.4). See also their formula (6.3).

(ii) There is an unexpected relation between $V'(0;\, b)$ and $L(b;\, b)$. From (2.11) and (2.12), we see that

$$V'(0;\, b) = \frac{r - s}{re^{rb} - se^{sb}}. \tag{3.9}$$

From (3.7) we gather that

$$L(b;\, b) = \frac{r - s}{re^{-sb} - se^{-rb}} = e^{(r+s)b}\,\frac{r - s}{re^{rb} - se^{sb}} = e^{(r+s)b}V'(0;\, b). \tag{3.10}$$

Noting that

$$r + s = -2\mu/\sigma^2, \tag{3.11}$$

we obtain from (3.10) the surprising identity

$$L(b;\, b) = e^{-2\mu b/\sigma^2}V'(0;\, b). \tag{3.12}$$

Now,

$$L(b;\, b) = \int_0^\infty e^{-\delta t}\Pr[t < T \le t + dt \,|\, X(0) = b], \tag{3.13}$$

$$V'(0;\, b) = \frac{d}{dx}\int_0^\infty e^{-\delta t}\mathrm{E}[I(T > t)[D(t + dt) - D(t)]\,|\,X(0) = x]\bigg|_{x=0}. \tag{3.14}$$

Here $I(A)$ denotes the indicator random variable of an event A. Because (3.12) is valid for all $\delta > 0$, it follows that

$$e^{2\mu b/\sigma^2}\,\Pr[t < T \le t + dt\,|\,X(0) = b] = \frac{d}{dx}\,\mathrm{E}[I(T > t)[D(t + dt) - D(t)]\,|\,X(0) = x]\bigg|_{x=0}. \tag{3.15}$$

In the appendix, we present the discrete counterparts of identities (3.12) and (3.15). Some readers will find that (A22) is easier to understand than (3.15).

(iii) Consider the limit $\sigma \to \infty$. Noting that r and s tend to 0, we gather from (2.12) that, for $\sigma \to \infty$,

$$g(x) \sim (r - s)x, \tag{3.16}$$

and

$$g'(x) \sim r - s. \tag{3.17}$$

Applying (3.16) and (3.17) to (2.11), we see that, for $0 \le x \le b$,

$$V(x;\, b) \;\to\; x \quad \text{as } \sigma \;\to\; \infty, \tag{3.18}$$

independently of b, μ and δ. Furthermore, the limit of (3.8) for $\sigma \to \infty$ is 0. Because T is a positive random variable, we conclude that its limiting distribution is the degenerate distribution at 0. Loosely speaking, the interpretation of these results is as follows: In the case of infinite risk, ruin is practically instantaneous, and the expectation of the dividends before ruin is equal to the initial surplus. That the latter depends neither on δ nor on μ is explained by the fact that ruin occurs "instantaneously." Formula (3.18) exhibits the limit of the *expectation* of the random variable D. In Section 4, more insight will be provided; we shall determine the limit of the *distribution* of D.

4. THE MOMENT-GENERATING FUNCTION OF *D*

If the barrier strategy with barrier level b is applied, the present value of the resulting dividends until ruin, D, is a random variable. Its expectation is given by (2.11). However, one might be interested in more detailed information concerning the distribution of D, for example, the higher order moments of D. This section examines the moment-generating function of D,

$$M(x, y;\, b) = \mathrm{E}[e^{yD}\,|\,X(0) = x]. \tag{4.1}$$

To obtain a functional equation for $M(x, y; b)$, assume $0 < X(0) = x < b$. Then

$$M(x, y;\, b) = \mathrm{E}[M(X(dt), e^{-\delta dt}y;\, b)]. \tag{4.2}$$

By expanding the last expression, we obtain, after simplification, the partial differential equation

$$\frac{\sigma^2}{2}\,\frac{\partial^2 M}{\partial x^2} + \mu\,\frac{\partial M}{\partial x} - \delta y\,\frac{\partial M}{\partial y} = 0, \tag{4.3}$$

which generalizes (2.7). Furthermore, the boundary conditions are

$$M(0, y;\, b) = 1 \tag{4.4}$$

and

$$\frac{\partial M(x, y;\, b)}{\partial x}\bigg|_{x=b} = yM(b, y;\, b), \tag{4.5}$$

which generalize (2.9) and (2.10), respectively.

To solve (4.3)–(4.5), we let

$$V_k(x;\, b) = \mathrm{E}[D^k], \qquad k = 1, 2, 3, \ldots \tag{4.6}$$

Note that $V_1(x; b) = V(x; b)$. Then,

$$M(x, y;\, b) = 1 + \sum_{k=1}^{\infty} \frac{y^k}{k!}\,\mathrm{E}[D^k] = 1 + \sum_{k=1}^{\infty} \frac{y^k}{k!}\,V_k(x;\, b), \tag{4.7}$$

substitution of which in (4.3) and comparing the coefficients of y^k yields the ordinary differential equations

$$\frac{\sigma^2}{2}\,V_k''(x;\, b) + \mu V_k'(x;\, b) - \delta k V_k(x;\, b) = 0, \qquad k = 1, 2, 3, \ldots \tag{4.8}$$

It follows from (4.4) that

$$V_k(0;\, b) = 0, \qquad k = 1, 2, 3, \ldots, \tag{4.9}$$

and from (4.5) that

$$V_1'(b;\, b) = 1, \tag{4.10}$$

which is (2.10), and that

$$V_k'(b;\, b) = kV_{k-1}(b;\, b), \qquad k = 2, 3, 4, \ldots. \tag{4.11}$$

From (4.8) and (4.9), it follows that, for $k = 1, 2, 3, \ldots,$

$$V_k(x;\, b) = C_k(b)\,g_k(x), \tag{4.12}$$

with

$$g_k(x) = e^{r_k x} - e^{s_k x}, \tag{4.13}$$

where r_k and s_k are the roots of the equation

$$\frac{\sigma^2}{2}\,\xi^2 + \mu\xi - \delta k = 0. \tag{4.14}$$

To determine the coefficient functions $C_k(\cdot)$, we apply (4.12) to (4.10) and to (4.11). We then obtain

$$C_1(b) = \frac{1}{g_1'(b)}, \tag{4.15}$$

which confirms (2.11), and

$$C_k(b)\,g_k'(b) = kC_{k-1}(b)\,g_{k-1}(b), \qquad k = 2, 3, 4, \ldots. \tag{4.16}$$

Hence,

$$C_k(b) = k!\,\frac{g_1(b)\cdots g_{k-1}(b)}{g_1'(b)\cdots g_{k-1}'(b)\,g_k'(b)}, \tag{4.17}$$

and the k-th moment of D about the origin is

$$V_k(x;\, b) = k!\,\frac{g_1(b)\cdots g_{k-1}(b)\,g_k(x)}{g_1'(b)\cdots g_{k-1}'(b)\,g_k'(b)}, \qquad k = 1, 2, 3, \ldots. \tag{4.18}$$

Finally, by (4.7), the moment-generating function of D is

$$\mathrm{E}[e^{yD}] = M(x, y; b) = 1 + \sum_{k=1}^{\infty} y^k \frac{g_1(b)\cdots g_{k-1}(b)g_k(x)}{g_1'(b)\cdots g_{k-1}'(b)g_k'(b)}. \tag{4.19}$$

Remark

For $\sigma \to \infty$, the limiting distribution of D can be determined as follows. Note that $r_k \to 0$ and $s_k \to 0$ for $\sigma \to \infty$. Hence, $g_k(x) \sim (r_k - s_k)x$ and $g_k'(x) \sim r_k - s_k$ for $\sigma \to \infty$. It follows from this and (4.19) that, for $\sigma \to \infty$,

$$\mathrm{E}[e^{yD}] \to 1 + \sum_{k=1}^{\infty} y^k b^{k-1} x = 1 + \frac{xy}{1-by} = \left(1 - \frac{x}{b}\right) + \frac{x}{b}\frac{1}{1-by}, \quad y < \frac{1}{b}. \tag{4.20}$$

Thus, the limiting distribution of D is a *mixture* of the degenerate distribution at 0 and the exponential distribution with mean b. The weights of this mixture, $(b - x)/b$ and x/b, are, respectively, the probability of not reaching and the probability of reaching the dividend barrier b before ruin; see (2.24). Formula (3.18) follows from (4.20).

5. THE DISTRIBUTION OF $D(T)$

Throughout this section, we consider $\delta = 0$ and $0 < x < b$. Hence, the functions $g(x)$ and $V(x; b)$ are given by (2.18) and (2.19), respectively. Because $D(T) = D$ when $\delta = 0$, the moment-generating function of $D(T)$ (which is the same as $X(T)$) can be obtained from (4.19) as the limiting case $\delta \to 0$. For simplicity, we shall not adjust our notation to signify that the interest rate is zero.

From (4.14) we see that $r_k = 0$, $s_k = -2\mu/\sigma^2$, and

$$g_k(x) = g(x) = 1 - e^{-2\mu x/\sigma^2} \tag{5.1}$$

for all k. Thus, we obtain from (4.19) and (2.11) that

$$\begin{aligned} M(x, y; b) = \mathrm{E}[e^{yD(T)}] &= 1 + \sum_{k=1}^{\infty} y^k [V(b; b)]^{k-1} V(x; b) \\ &= 1 + \frac{V(x; b)y}{1 - V(b; b)y} = \left[1 - \frac{V(x; b)}{V(b; b)}\right] + \frac{V(x; b)}{V(b; b)}\frac{1}{1 - V(b; b)y}. \end{aligned} \tag{5.2}$$

This shows that the distribution of $D(T)$ is a mixture of the degenerate distribution at 0 and the exponential distribution with mean $V(b; b)$. The weights of this mixture are

$$p = 1 - \frac{V(x; b)}{V(b; b)} = 1 - \frac{g(x)}{g(b)} \tag{5.3}$$

and

$$q = 1 - p = \frac{V(x; b)}{V(b; b)} = \frac{g(x)}{g(b)}, \tag{5.4}$$

respectively. Note that p is the probability that the surplus does not reach the barrier b before ruin occurs. With $\delta = 0$, $V(b; b)$ is the expectation $\mathrm{E}[D(T)|X(0) = b]$.

By considering the visits of the modified surplus at the dividend barrier b that are separated by visits at the initial level x, we see that $D(T)$ has a compound geometric distribution,

$$D(T) = D_1 + D_2 + \cdots + D_N. \tag{5.5}$$

Here, D_j is the dividends paid between visits j and $j + 1$ at the initial level x. We can rewrite (5.2) as

$$M(x, y; b) = p \sum_{n=0}^{\infty} \left(\frac{q}{1 - pV(b; b)y} \right)^n .$$

This confirms that $D(T)$ has a compound geometric distribution, and it shows that

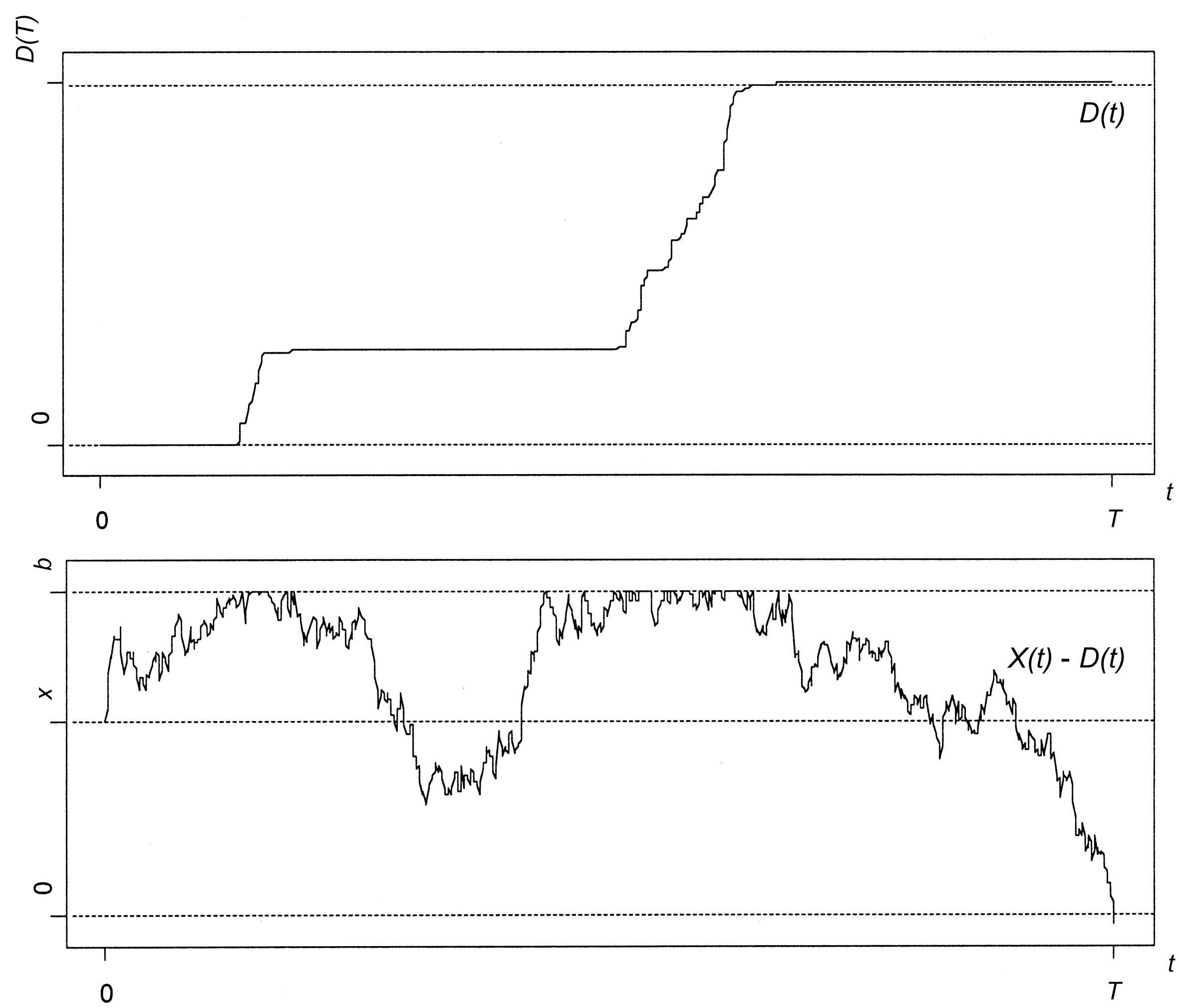
$$\mathrm{E}[N] = \frac{q}{p} = \frac{g(x)}{g(b) - g(x)} \tag{5.6}$$

and that the common distribution of the D_j's is exponential with mean

$$pV(b; b) = V(b; b) - V(x; b) = V(b - x; b - x) \tag{5.7}$$

by (5.3). The decomposition of $D(T)$ is illustrated in Figure 2, where $N = 2$.

Figure 2
The Decomposition of $D(T)$ as a Compound Geometric Random Variable



Remarks

(i) As $x \uparrow b$, it follows from (5.3) that $p \to 0$ and, thus, the number of visits N becomes infinite. Also, we see from (5.7) that $E[Dj] \to 0$.

(ii) As $x \downarrow 0$, we have $p \to 1$, N becomes zero, and $D(T)$ is the zero random variable. Nevertheless, the D_j's have a limiting distribution: It is the exponential distribution with mean $V(b; b)$; see (5.7).

(iii) Consider $\mu \to 0$ (in addition to $\delta = 0$). Applying (2.22) to the second to the last expression in (5.2) yields

$$M(x, y; b) = 1 + \frac{xy}{1 - by}. \tag{5.8}$$

Note that the right-hand side of (5.8) does not involve σ and, more intriguingly, it is identical to the second to the last expression in (4.20), which was obtained by letting $\sigma \to \infty$. This is not a coincidence, and we can explain it by means of *operational time.* Let

$$\tilde{t} = \sigma^2 t. \tag{5.9}$$

In the terms of the new time scale, the parameters of the model are:

$$\tilde{\sigma} = 1, \qquad \tilde{\mu} = \frac{\mu}{\sigma^2}, \qquad \tilde{\delta} = \frac{\delta}{\sigma^2}.$$

Thus $\sigma \to \infty$ means that $\tilde{\mu} \to 0$ and $\tilde{\delta} \to 0$.

6. THE OPTIMAL DIVIDEND BARRIER

For a given initial surplus $X(0) = x$, let b^* denote the optimal value of b, that is, the value that maximizes $V(x; b)$, the expectation of D. From (2.11), we see that this is the value minimizing $g'(b)$. Hence, b^* is the solution of the equation

$$g''(b^*) = 0. \tag{6.1}$$

This leads to

$$b^* = \frac{1}{r - s} \ln\left(\frac{s^2}{r^2}\right) = \frac{2}{r - s} \ln\left(\frac{-s}{r}\right), \tag{6.2}$$

with r and s given by (2.14) and (2.15), respectively. Note that the optimal barrier level b^* does not depend on the initial surplus x. Formula (6.2) can be found in Gerber (1972) and viewed as a limiting case of formula (1.15) in Gerber (1979, p. 149) and of (7.10) in Gerber and Shiu (1998).

The optimal value of b has a geometric characterization. Let $W(x; b)$, $x > 0$, be the expectation of D if the barrier strategy with parameter b is applied. Thus,

$$W(x; b) = \begin{cases} V(x; b) & \text{if } 0 \le x \le b \\ x - b + V(b; b) & \text{if } x > b \end{cases}. \tag{6.3}$$

At the junction $x = b$, the function $W(x; b)$ is continuous and has a continuous first derivative by (2.10). Under what condition is also the second derivative continuous, that is, $V''(b; b) = 0$? From (2.11), we see that

$$V''(x; b) = \frac{g''(x)}{g'(b)}. \tag{6.4}$$

Hence, $V''(b; b) = 0$ is equivalent to the condition that $g''(b) = 0$, which in turn means that $b = b^*$ by (6.1). This geometric characterization of the optimal parameter value is known as a *high contact condition* in the finance literature and a *smooth pasting condition* in the optimal stopping literature.

With $b = b^*$ (and V defined by equation 2.11), (6.3) is Theorem 3.2 of Asmussen and Taksar (1997) and Theorem 4.2 of Højgaard and Taksar (1999). (The first two plus signs in Theorem 4.2 of Asmussen and Taksar (1997) should be changed to minus signs.) A main tool of theirs is *stochastic control theory* (*Hamilton-Jacobi-Bellman equation*). Further discussion on the smooth pasting condition can also be found in Asmussen and Taksar (1997).

7. ANALYSIS OF THE OPTIMAL BARRIER

If the initial surplus is at the optimal barrier, $x = b = b^*$, the differential equation (2.7) becomes

$$\frac{\sigma^2}{2} 0 + \mu 1 - \delta V(b^*;\, b^*) = 0$$

because of the condition $V''(b^*;\, b^*) = 0$ and condition (2.10). Hence,

$$V(b^*;\, b^*) = \frac{\mu}{\delta}. \tag{7.1}$$

This formula has been obtained by Gerber (1972). It can also be found in Jeanblanc-Picqué and Shiryaev (1995, eq. 2.37). At first sight, (7.1) is a surprising result, since μ/δ does not depend on σ and is identical to the present value of a perpetuity-certain with continuous payments at a rate of μ. In the deterministic case $\sigma = 0$ (that is, $X(t) = x + \mu t$), ruin does not occur and, for each $b > 0$, $V(b, b) = \mu/\delta$. However, for the stochastic case $\sigma > 0$, ruin does occur and the dividends stop at the time of ruin.

For a better understanding of (7.1), observe that b^* is the initial surplus necessary to obtain an expected total return of μ/δ. Since the latter is independent of σ, the necessary initial surplus b^* must be a function of σ to compensate for the risk. In fact, we now show that

(I) b^* is an increasing function of σ; (7.2)

(II) $b^* \uparrow \dfrac{\mu}{\delta}$ for $\sigma \uparrow \infty$; and (7.3)

(III) $b^* \downarrow 0$ for $\sigma \downarrow 0$. (7.4)

The meaning of statement (III) is that, if the business has no risk, there is no need for the company to hold any surplus. Statement (II) is compatible with (3.18) and (7.1). From statement (II) and equation (7.1), we see that $b^* < V(b^*;\, b^*)$, which is a special case of the inequality $x < V(x;\, b^*)$, for $0 < x \le b^*$. The latter can be reasoned as follows: If the initial surplus is x, applying the barrier strategy with parameter b^* is better than paying out the amount x immediately with instantaneous ruin. We shall in fact show in Section 8 that

$$x < V(x;\, b) \qquad \text{for } 0 < x \le b \le b^*. \tag{7.5}$$

To prove these three statements, we need an alternative form for (6.2). Motivated by

$$\frac{-s}{r} = \frac{\sqrt{\mu^2 + 2\delta\sigma^2} + \mu}{\sqrt{\mu^2 + 2\delta\sigma^2} - \mu},$$

which is derived from applying formulas (2.14) and (2.15), we introduce a new variable

$$z = \frac{\mu}{\sqrt{\mu^2 + 2\delta\sigma^2}}, \tag{7.6}$$

so that

$$\frac{-s}{r} = \frac{1+z}{1-z}. \tag{7.7}$$

Note that z is a decreasing function of σ, and that $z = 0$ for $\sigma = \infty$, and $z = 1$ for $\sigma = 0$. Now, (6.2) becomes

$$b^* = \frac{\sigma^2}{\mu} z \ln \frac{1+z}{1-z} \tag{7.8}$$

$$= \frac{\mu}{2\delta}\left(\frac{1}{z} - z\right) \ln \frac{1+z}{1-z}. \tag{7.9}$$

For $0 \le z < 1$, we have

$$\ln \frac{1+z}{1-z} = \ln(1+z) - \ln(1-z) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots\right).$$

Hence,

$$b^* = \frac{\mu}{\delta}\left(1 - \frac{2}{3}z^2 - \frac{2}{15}z^4 - \cdots - \frac{2}{(2n-1)(2n+1)}z^{2n} - \cdots\right). \tag{7.10}$$

We see from (7.10) that b^* is a decreasing function of z, $0 \le z < 1$, and that $b^* \uparrow \mu/\delta$ for $z \downarrow 0$, proving (7.2) and (7.3). Furthermore,

$$\frac{2}{3} + \frac{2}{15} + \cdots + \frac{2}{(2n-1)(2n+1)} + \cdots = 1. \tag{7.11}$$

To verify (7.11), use

$$\frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}$$

to write its left-hand side as a telescoping series. It now follows from (7.10) and (7.11) that $b^* \downarrow 0$ for $z \uparrow 1$, proving (7.4).

8. THE LEVERAGE RATIO

For $0 < x \le b \le b^*$, consider the ratio

$$R(x; b) = \frac{V(x; b)}{x}, \tag{8.1}$$

which is the expected present value of all dividends per unit of initial capital or surplus. Because $V(x; b)/x = [V(x; b) - V(0; b)]/(x - 0)$, we see that $R(x; b)$ can be interpreted as the slope of a secant. Thus, $R(0; b)$ is defined as the derivative $V'(0; b)$. Since $g(x)$ is a concave function for $x \le b^*$, it follows from (2.11) that, for $0 \le x \le b \le b^*$, $V(x; b)$ is also a concave function of x. Hence, $R(x; b)$ is a decreasing function of x, $0 \le x \le b \le b^*$, and

$$R(b; b) > V'(b; b) = 1 \tag{8.2}$$

by (2.11). The inequality $R(x; b) > 1$ is equivalent to inequality (7.5). Also, the fact that $R(x; b)$ is a decreasing function of the initial capital x has a somewhat shocking implication: If the investor is only interested in the leverage ratio, he or she would want to invest in companies with a low degree of capitalization!

For the remainder of this section, we consider $b = b^*$, the optimal barrier level, and $0 \le x \le b^*$. The minimum leverage ratio is

$$R(b^*; b^*) = \frac{V(b^*; b^*)}{b^*} = 2\left[\left(\frac{1}{z} - z\right) \ln \frac{1+z}{1-z}\right]^{-1}$$

$$= \left(1 - \frac{2}{3}z^2 - \frac{2}{15}z^4 - \cdots - \frac{2}{(2n-1)(2n+1)}z^{2n} - \cdots\right)^{-1} \tag{8.3}$$

by (7.1), (7.9), and (7.10), with z given by (7.6). Obviously, $R(b^*; b^*) > 1$, confirming (8.2). On the other hand, the maximum leverage ratio is

$$R(0; b^*) = V'(0; b^*) = \frac{r - s}{re^{rb^*} - se^{sb^*}} \tag{8.4}$$

by (3.9). Substituting b^* in (8.4) by (6.2) yields

$$R(0; b^*) = \frac{r - s}{r(-s/r)^{2r/(r-s)} - s(-s/r)^{2s/(r-s)}}. \tag{8.5}$$

Now, the ratio $-s/r$ is given by (7.7). Also, it follows from (7.7) that $r/(r - s) = (1 - z)/2$ and that $s/(r - s) = -(1 + z)/2$. Applying these to (8.7), we obtain

$$R(0; b^*) = \frac{2}{(1-z)\left(\frac{1+z}{1-z}\right)^{1-z} + (1+z)\left(\frac{1+z}{1-z}\right)^{-(1+z)}} = \left(\frac{1+z}{1-z}\right)^{z}. \tag{8.6}$$

From (8.3) and (8.6), we see that both the minimal and the maximal leverage ratios, $R(b^*; b^*)$ and $R(0; b^*)$, are increasing functions of z, $0 \le z < 1$. Rewriting (7.6) as

$$z = \frac{1}{\sqrt{1 + 2\delta(\sigma/\mu)^2}}, \tag{8.7}$$

we see that both $R(b^*; b^*)$ and $R(0; b^*)$ are decreasing functions of both δ and the "coefficient of variation" σ/μ.

Formulas (8.8) and (8.9) show that $R(0; b^*) \to 1$ as $\sigma/\mu \to \infty$. Now, $R(0; b^*) \ge R(x; b^*)$ for $0 \le x \le b^*$. Recall from (7.3) that $b^* \to \mu/\delta$ as $\sigma \to \infty$. Hence, given $\mu > 0$ and $\delta > 0$, we have $R(x; b^*) \to 1$ as $\sigma \to \infty$ for $0 \le x \le \mu/\delta$. In the case of infinite risk, it is not possible to have an expected total return exceeding the initial capital.

On the other hand, $R(b^*; b^*) = (\mu/\delta)/b^*$ by (7.1). Recall from (7.4) that $b^* \to 0$ as $\sigma \to 0$. Hence, as $\sigma \to 0$, $R(b^*; b^*) \to \infty$ and $R(0; b^*) \to \infty$. If the business has no risk, the leverage ratio becomes infinite.

Remark

A special case of (3.12) is

$$L(b^*; b^*) = e^{-2\mu b^*/\sigma^2} V'(0; b^*) = e^{-2\mu b^*/\sigma^2}\left(\frac{1+z}{1-z}\right)^{z} \tag{8.8}$$

by (8.4) and (8.6). Substituting b^* on the right-hand side of (8.8) by (7.8) yields

$$L(b^*; b^*) = \left(\frac{1+z}{1-z}\right)^{-2z}\left(\frac{1+z}{1-z}\right)^{z} = \left(\frac{1-z}{1+z}\right)^{z}, \tag{8.9}$$

with z given by (8.7). From this and (8.6), we see that

$$R(0;\, b^*)L(b^*;\, b^*) = 1. \tag{8.10}$$

This identity does not seem to have an apparent interpretation.

9. THE IMPLIED UTILITY FUNCTION

The value of an initial capital of x is $V(x;\, b^*)$, $0 \le x \le b^*$. Thus, $V(x;\, b^*)$ can be interpreted as some sort of a utility of x. We shall show that $V(x;\, b^*)$, indeed, has the properties of a risk-averse utility function. In view of (2.11), we examine the function

$$g(x) = e^{rx} - e^{sx}, \qquad 0 \le x \le b^*, \tag{9.1}$$

for its properties as a utility function. Note that

$$g'(x) = re^{rx} - se^{sx} > 0.$$

The *implied risk aversion function* is

$$\zeta(x) = \frac{-g''(x)}{g'(x)} = \frac{-r^2e^{rx} + s^2e^{sx}}{re^{rx} - se^{sx}}. \tag{9.2}$$

Observe that

$$\zeta'(x) = \frac{rs(r-s)^2e^{rx+sx}}{(re^{rx} - se^{sx})^2} \tag{9.3}$$

after simplification. Because $rs = -2\delta/\sigma^2 < 0$, it follows that $\zeta(x)$ is a strictly decreasing function of x. As $\zeta(b^*) = 0$ by (6.1), we have $\zeta(x) > 0$ for $0 \le x < b^*$. Furthermore, we note that

$$\zeta(0) = -(r+s) = \frac{2\mu}{\sigma^2} \tag{9.4}$$

by (9.2) and (3.11).

10. THE EXPECTED UTILITY OF *D*

One way to take into account the randomness of D is to calculate its expected utility. Let $u(x)$ be an appropriate risk-averse utility function. By the term “risk-averse,” we mean that the function has the properties $u'(x) > 0$ and $u''(x) < 0$. Examples are the quadratic utility function with level of saturation s,

$$u(x) = x - \frac{1}{2s}x^2, \qquad x < s, \tag{10.1}$$

and the exponential utility function with parameter $\zeta > 0$,

$$u(x) = \frac{1}{\zeta}(1 - e^{-\zeta x}), \qquad -\infty < x < \infty. \tag{10.2}$$

We are interested in the expected utility, $\mathrm{E}[u(D)]$.

If the barrier strategy with level b is applied, the expected utility for (10.1) and (10.2) can be calculated as follows. For the quadratic utility function (10.1), we obtain

$$\mathrm{E}[u(D)] = \mathrm{E}[D] - \frac{1}{2s}\mathrm{E}[D^2] = V_1(x;\, b) - \frac{1}{2s}V_2(x;\, b) = \frac{g_1(x)}{g_1'(b)} - \frac{1}{s}\frac{g_1(b)g_2(x)}{g_1'(b)g_2'(b)}, \tag{10.3}$$

according to (4.18). For the exponential utility function (10.2), we get

$$\mathrm{E}[u(D)] = \frac{1}{\zeta}(1 - \mathrm{E}[e^{-\zeta D}]) = \frac{1}{\zeta}(1 - M(x, -\zeta; b)) = \sum_{k=1}^{\infty} (-\zeta)^{k-1} \frac{g_1(b)\cdot \cdot \cdot g_{k-1}(b) g_k(x)}{g_1'(b)\cdot \cdot \cdot g_{k-1}'(b) g_k'(b)}, \tag{10.4}$$

according to (4.19).

This leads us to the following question: What is the optimal dividend strategy if the objective is to maximize the expected utility of the present value of the dividends until ruin? If the utility function is quadratic or exponential, the optimal strategy cannot be a barrier strategy, because, if it were, we would maximize expressions (10.3) or (10.4), respectively, and the optimal value of b would not depend on x. But this is obviously not the case. Hence, maximizing the expected utility of the present value of the dividends until ruin is a new and challenging problem.

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APPENDIX

The goal of this appendix is to derive the discrete counterparts of identities (3.12) and (3.15) for the original De Finetti (1957) model. In this model, the surplus at time t is

$$X(t) = x + G_1 + \cdot \cdot \cdot + G_t \tag{A1}$$

$t = 1, 2, 3, \ldots$. Here $x = X(0)$ is the initial surplus (a positive integer), and the annual net gains G_1, $G_2, \ldots$ are independent and identically distributed random variables with

$$\Pr(G_t = 1) = \pi,\ \Pr(G_t = -1) = 1 - \pi, \tag{A2}$$

where $\pi > \frac{1}{2}$. Let

$$M(t) = \max\{x, X(1), \ldots, X(t)\}. \tag{A3}$$

The dividend barrier b is a positive integer, $b \geq x$. Then the cumulative dividends by time t are given by

$$D(t) = \begin{cases} 0 & \text{if } M(t) \leq b, \\ M(t) - b & \text{if } M(t) > b. \end{cases} \tag{A4}$$

Let d_t denote the dividend paid at time t. Thus,

$$D(t) = d_1 + d_2 + \cdot \cdot \cdot + d_t. \tag{A5}$$

Note that the random variable d_t assumes only the values 0 or 1. Let $0 < v < 1$ be a discount factor,

$$T = \min\{t \geq 0 \mid X(t) - D(t) = 0\} \tag{A6}$$

be the time of ruin, and

$$V(x;\, b) = \mathrm{E}\left[\sum_{t=1}^{T} v^t d_t \,\middle|\, X(0) = x\right] \tag{A7}$$

be the expectation of the present value of the dividends until ruin.

Let $x = 1, 2, \ldots, b - 1$. By distinguishing whether $G_1 = 1$ or $G_1 = -1$, we see that

$$V(x;\, b) = v\{\pi V(x + 1;\, b) + (1 - \pi)V(x - 1;\, b)\}. \tag{A8}$$

Thus, as a function of x, $V(x;\, b)$ is a linear combination of r^x and s^x, where $0 < s < 1 < r$ are the solutions of the indicial equation

$$v\pi\zeta^2 - \zeta + v(1 - \pi) = 0. \tag{A9}$$

The coefficients of this linear combination are determined from the boundary conditions

$$V(0;\, b) = 0 \tag{A10}$$

and

$$V(b;\, b) = v\{\pi[1 + V(b;\, b)] + (1 - \pi)V(b - 1;\, b)\}. \tag{A11}$$

It follows that

$$V(x;\, b) = \frac{r^x - s^x}{(r - 1)r^b - (s - 1)s^b}, \tag{A12}$$

$x = 0, 1, \ldots, b$. In particular,

$$V(1;\, b) = \frac{r - s}{(r - 1)r^b - (s - 1)s^b}. \tag{A13}$$

Let

$$L(x;\, b) = \mathrm{E}[v^T | X(0) = x] \tag{A14}$$

denote the expected discounted value of a payment of 1 at the time of ruin. As a function of x, $L(x;\, b)$ satisfies a difference equation like (A8). Hence, $L(x;\, b)$ is also a linear combination of r^x and s^x. The boundary conditions are now

$$L(0;\, b) = 1 \tag{A15}$$

and

$$L(b;\, b) = v\{\pi L(b;\, b) + (1 - \pi)L(b - 1;\, b)\}. \tag{A16}$$

It follows that

$$L(x;\, b) = \frac{(r - 1)s^{-(b-x)} - (s - 1)r^{-(b-x)}}{(r - 1)s^{-b} - (s - 1)r^{-b}}, \tag{A17}$$

$x = 0, 1, \ldots, b$. In particular,

$$L(b;\, b) = \frac{r - s}{(r - 1)s^{-b} - (s - 1)r^{-b}} = (rs)^b \,\frac{r - s}{(r - 1)r^b - (s - 1)s^b} = (rs)^b V(1;\, b) \tag{A18}$$

by (A13). From (A9) we gather that $rs = (1 - \pi)/\pi$. Hence,

$$L(b;\, b) = \left(\frac{1 - \pi}{\pi}\right)^b V(1;\, b), \tag{A19}$$

which is the counterpart of (3.12). Now, it follows from (A14) and (A7) that

$$L(x;\ b) = \sum_{t=1}^{\infty} v^t \ \Pr(T = t | X(0) = x) \tag{A20}$$

and

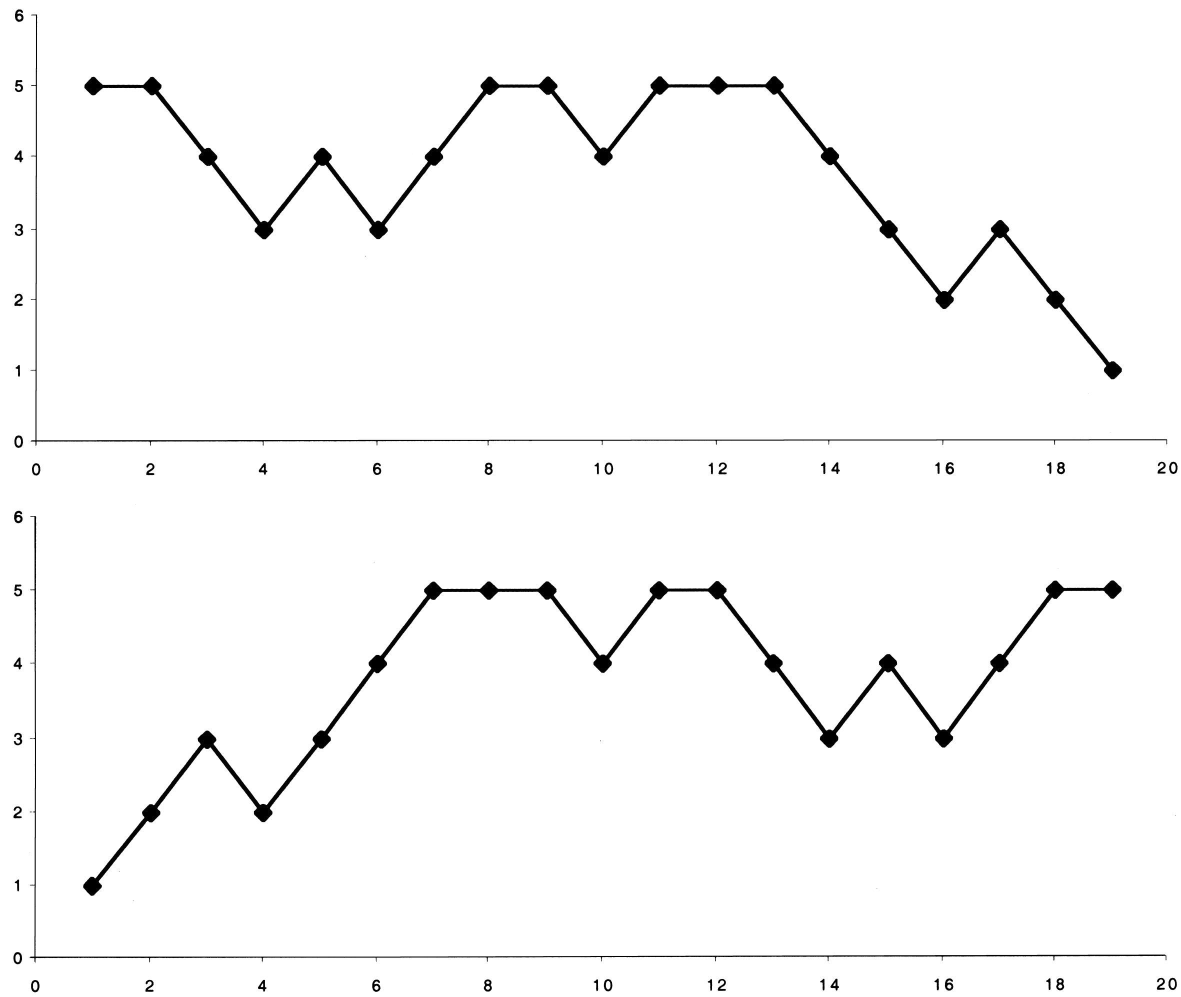
$$V(x;\ b) = \sum_{t=1}^{\infty} v^t \ \Pr(d_t = 1 \text{ and } T > t | X(0) = x). \tag{A21}$$

Because (A19) holds for all $v \in (0, 1)$, we conclude that, for $t = 1, 2, 3, \ldots,$

$$\Pr(T = t | X(0) = b) = \left(\frac{1 - \pi}{\pi}\right)^b \Pr(d_t = 1 \text{ and } T > t | X(0) = 1). \tag{A22}$$

The probabilistic identities (A22) correspond to (3.15).

Figure 3
Duality of the Sample Paths



Formula (A22) can be proved directly using the notion of *duality,* which, as pointed out by Feller (1971), enables one "to prove in an elementary way theorems that would otherwise require deep analytic methods" (p. 395). First observe that (A22) is equivalent to

$$\Pr(X(t-1)=1 \text{ and } T>t-1 \,|\, X(0)=b) = \left(\frac{1-\pi}{\pi}\right)^{b-1} \Pr(X(t-1)=b \text{ and } T>t-1 \,|\, X(0)=1). \quad \text{(A23)}$$

There is a one-to-one correspondence between (a) the sample paths that contribute to the left-hand side of (A23) and (b) the sample paths that contribute to the right-hand side of (A23): For a given sample path of the type (a), there is a dual sample path of type (b), which is obtained by a reversal of the time axis. This is illustrated in Figure 3, where $b = 5$, $t = 20$, $d = 4$, $m = 5$, and $n = 9$. Now consider a sample path of type (a). Let d be the total dividends by time $t - 1$, m the number of gains of $+1$ when the surplus at the beginning of the period is less than b, and n the number of gains of -1 by time $t - 1$. We must have $n - m = b - 1$. Hence, the probability of such a sample path is

$$\pi^{d+m}(1-\pi)^n = \pi^{d+n-(b-1)}(1-\pi)^{m+(b-1)}. \quad \text{(A24)}$$

For the dual sample path, the total dividends are also d, the number of gains of $+1$ when the surplus at the beginning of the period is less than b is now n, and m is now the number of gains of -1. Thus, the probability of the dual sample path is

$$\pi^{d+n}(1-\pi)^m. \quad \text{(A25)}$$

Hence, the probability of a given sample path of type (a) is $[(1-\pi)/\pi]^{b-1}$ times the probability of the dual sample path. This proves (A23).

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Discussions on this paper can be submitted until July 1, 2004. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.

comparison to the GPD. In particular, the approximation error of the new model is of order $o(A(u))$, while the Pareto model ($\delta = 0$) and the GPD ($\rho = -1$) both have an approximation error of order $O(A(u))$.

CONCLUSION

Extreme-value theory justifies a one-parameter extension of the classical GPD model for excesses over a high threshold. The new model gives an accurate approximation of the true excess distribution for much lower thresholds than the GPD is capable of. For the SOA's Large Claims Database, the model provides an adequate description of the complete data set of large claims.

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"Optimal Dividends: Analysis with Brownian Motion" Hans U. Gerber and Elias S.W. Shiu, January 2004

OLIVIER DEPREZ*

I enjoyed reading this paper, which treats a difficult mathematical topic in an accessible language. The paper brought back positive memories of my days at the university. The reader appreciates that, in many cases, the paper presents the chicken (the algebraic derivation of a formula) in conjunction with the egg (the interpretation of the formula).

When I saw formula (2.21) of the paper and read its interpretation, it occurred to me that this formula could be generalized in two ways: (1) in the case where the initial surplus is not on the barrier, and (2) in the case where the force of interest is positive. Let $0 < x < y \le b$, and suppose $X(0) = y$. Because of the continuous trajectories of the surplus process, the dividends can be decomposed as those before the surplus ever drops to the level x and those afterwards. Hence, by interpretation,

$$V(y;\, b) = V(y - x;\, b - x) + L(y - x;\, b - x) \cdot V(x;\, b),$$

where L is defined in formula (3.1) of the paper. Thus,

$$L(y - x;\, b - x) = \frac{V(y;\, b) - V(y - x;\, b - x)}{V(x;\, b)}.$$

This is another way to calculate the L function, starting with formulas (2.11) and (2.12) for the V function.

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HANSJÖRG ALBRECHER*

This paper by Professors Gerber and Shiu is an interesting contribution to the field and provides valuable additional insight into the problem of optimizing dividend payments. I would like to add two comments:

In Section 4, the moment-generating function $M(x, y; b) = \mathbb{E}(e^{yD}|X(0) = x)$ of the present value D of the dividend payments until ruin is determined. The same technique can be applied to obtain $M(x, y; b)$ in the classical model of risk theory. Here, the aggregate claims constitute a compound Poisson process, say with claim frequency λ and individual claim amount distribution $F(z)$. Then, by conditioning on the occurrence of a claim (and its amount) in the interval from 0 to dt, we see that for $0 < x < b$,

$$M(x, y;\, b) = (1 - \lambda\, dt)M(x + c\, dt, ye^{-\delta\, dt};\, b) + \lambda\, dt \int_0^{x+c\,dt} M(x + c\, dt - z, ye^{-\delta\, dt};\, b)\, dF(z) + \lambda\, dt \int_{x+c\,dt}^{\infty} dF(z) + o(dt)$$

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$$= (1 - \lambda\, dt) M(x + c\, dt, ye^{-\delta\, dt}; b) + \lambda\, dt \int_0^x M(x - z, y; b)\, dF(z) + \lambda\, dt \int_x^\infty dF(z) + o(dt).$$

Taylor expansion and collecting all the terms of order dt yields

$$c \frac{\partial M(x, y; b)}{\partial x} - \delta y \frac{\partial M(x, y; b)}{\partial y} - \lambda M(x, y; b) + \lambda \int_0^x M(x - z, y; b)\, dF(z) + \lambda(1 - F(x)) = 0. \quad (1)$$

Similarly, for $x = b$ we have

$$M(b, y; b) = (1 - \lambda\, dt) e^{yc\, dt} M(b, ye^{-\delta\, dt}; b) + \lambda\, dt \int_0^b e^{yc\, dt} M(b - z, ye^{-\delta\, dt}; b)\, dF(z) + \lambda\, dt \int_b^\infty e^{yc\, dt}\, dF(z) + o(dt)$$
$$= (1 - \lambda\, dt) e^{yc\, dt} M(b, ye^{-\delta\, dt}; b) + \lambda\, dt \int_0^b M(b - z, y; b)\, dF(z) + \lambda\, dt \int_b^\infty dF(z) + o(dt),$$

from which it follows that

$$cyM(b, y; b) - \delta y \frac{\partial M(b, y; b)}{\partial y} - \lambda M(b, y; b) + \lambda \int_0^b M(b - z, y; b)\, dF(z) + \lambda(1 - F(b)) = 0.$$

Setting $x = b$ in formula (1) and comparing the resulting formula with the last formula, we obtain the boundary condition

$$\left. \frac{\partial M(x, y; b)}{\partial x} \right|_{x=b} = yM(b, y; b), \quad (2)$$

which corresponds to (4.5) of the paper. Using the representation (4.7) of the paper in (1) and (2) here, and equating the coefficients of y^k, directly yields the integro-differential equations

$$c \frac{\partial V_k(x, b)}{\partial x} - (\lambda + k\delta) V_k(x, b) + \lambda \int_0^x V_k(x - z, b)\, dF(z) = 0 \quad (k = 1, 2, \ldots), \quad (3)$$

together with the boundary conditions

$$\left. \frac{\partial V_k(x, b)}{\partial x} \right|_{x=b} = kV_{k-1}(b, b) \quad (k = 1, 2, \ldots), \quad (4)$$

where $V_0(b, b) = 1$. Equations (3) together with (4), for the moments $V_k(x, b)$ of D in the classical risk model with constant dividend barrier coincide with (2.1) and (2.2) of Dickson and Waters (2004), who recently derived this result using a different technique. Note that the above approach is particularly simple.

One might wonder whether there is an intuitive reason for the fact that the k-th moment of D depends on lower moments only through the $(k - 1)$-th moment (cf. (4) and correspondingly condition (4.11) of the paper). For that purpose, consider a direct derivation of $V_k(x, b)$ in the classical risk model by the differential argument: For $0 < x < b$, we have

$$V_k(x, b) = (1 - \lambda\, dt) e^{-k\delta\, dt} V_k(x + c\, dt, b) + e^{-k\delta\, dt} \lambda\, dt \int_0^{x + c\, dt} V_k(x + c\, dt - z, b)\, dF(z) + o(dt)$$
$$= (1 - \lambda\, dt) e^{-k\delta\, dt} V_k(x + c\, dt, b) + \lambda\, dt \int_0^x V_k(x - z, b)\, dF(z) + o(dt),$$

from which we obtain equation (3) by expansion. Now, for $x = b$,

$$V_k(b, b)$$
$$= (1 - \lambda\, dt)\mathrm{E}[(c\, dt + e^{-\delta\, dt}D)^k | X(0) = b]$$
$$+ \lambda\, dt \int_0^b \mathrm{E}[(c\, dt + e^{-\delta\, dt}D)^k | X(0) = b - z]\, dF(z)$$
$$+ \lambda\, dt \int_b^\infty (c\, dt)^k\, dF(z) + o(dt)$$
$$= (1 - \lambda\, dt)\mathrm{E}[(c\, dt + e^{-\delta\, dt}D)^k | X(0) = b]$$
$$+ \lambda\, dt \int_0^b \mathrm{E}[D^k | X(0) = b - z]\, dF(z) + o(dt),$$

so that, by the binomial formula, we see that only the $(k - 1)$-th and the k-th moment contribute to the significant terms of order dt:

$$V_k(b, b) = kc\, dt V_{k-1}(b, b)$$
$$+ (1 - k\delta\, dt)V_k(b, b) - \lambda\, dt V_k(b, b)$$
$$+ \lambda\, dt \int_0^b V_k(b - z, b)\, dF(z) + o(dt),$$

from which we get

$$kcV_{k-1}(b, b) - (\lambda + k\delta)V_k(b, b)$$
$$+ \lambda \int_0^b V_k(b - z, b)\, dF(z) = 0,$$

and equation (4) finally follows by continuity.

My second comment refers to Section 10, where the authors point out the need to study the maximization of the expected utility of the sum of discounted dividend payments until ruin. A related problem is to try to maximize the expectation of the sum of the discounted utilities of the dividend payments until ruin instead (note that for linear utility functions the two problems coincide, but in general they are different). Hubalek and Schachermayer (2004) investigated the latter problem in detail for power utility functions by means of the corresponding Hamilton-Jacobi-Bellman equation, and in that case it turns out that the optimal dividend strategy is indeed not of barrier type.

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AUTHORS' REPLY

We are grateful to receive these two insightful discussions. Dr. Deprez's elegant observation is in the time-honored tradition of "general reasoning" in actuarial science. Dr. Albrecher has shown that the boundary conditions at the barrier are the same in the compound Poisson model as in the Brownian motion model; see formulas (2.10), (4.5) and (4.11) in the paper, and (2) and (4) in his discussion. We shall show how these conditions can be obtained by a unified and general approach. This approach is intuitive and simple, but it might not satisfy a mathematical purist. We assume that the surplus process of the company is a Lévy process without any upward jumps. For a given nonnegative, differentiable function $w(z)$, $z \geq 0$, we define

$$\Phi(x, y; b) = \mathrm{E}[w(yD) | X(0) = x], \quad 0 \leq x \leq b, \tag{R1}$$

where D is the present value of all dividends until ruin. We claim that the following boundary condition holds:

$$\left.\frac{\partial \Phi(x, y; b)}{\partial x}\right|_{x=b} = y\mathrm{E}[w'(yD) | X(0) = b]. \tag{R2}$$

Note that (4.11) and (4) are the special case with $w(z) = z^k$ (and $y = 1$), and (4.5) and (2) are the special case with $w(z) = e^z$.

To obtain (R2), we use the notation $D = D_x$ if $X(0) = x$ and consider two situations, $X(0) = b$ and $X(0) = b - h$ with h positive and "small." Then the following approximate relation holds between the random variables D_b and D_{b-h}:

$$D_b \approx h + D_{b-h}.$$

Hence,

$$w(yD_b) - w(yD_{b-h}) \approx yhw'(yD_b),$$

taking expectations of which yields

$$\Phi(b, y; b) - \Phi(b - h, y; b) \approx yh\mathrm{E}[w'(yD_b)].$$

Now, we divide both sides by h and let $h \to 0$ to obtain (R2).

In the compound Poisson model, the function $\Phi(x, y; b)$ satisfies for $0 < x < b$ the equation

$$c\,\frac{\partial\Phi(x, y; b)}{\partial x} - \delta y\,\frac{\partial\Phi(x, y; b)}{\partial y} - \lambda\Phi(x, y; b) + \lambda\int_0^x \Phi(x - z, y; b)\, dF(z) = -\lambda w(0)[1 - F(x)], \qquad \text{(R3)}$$

which generalizes Albrecher's equation (1). In fact, (R3) can also be obtained by the *infinitesimal method*. Alternatively, consider a time interval from 0 to h, $0 < h < (b - x)/c$. Then, by conditioning on the time and amount of the first claim in this interval, we see that

$$\begin{aligned}\Phi(x, y; b) &= e^{-\lambda h}\Phi(x + ch, e^{-\delta h}y; b)\\ &\quad + \lambda\int_0^h e^{-\lambda t}\int_0^{x+ct}\Phi(x + ct - z, e^{-\delta t}y; b)dF(z)dt\\ &\quad + \lambda\int_0^h e^{-\lambda t}w(0)[1 - F(x + ct)]\, dt.\end{aligned}$$

Differentiating the above with respect to h, setting $h = 0$, and rearranging, we obtain (R3).

In the Brownian motion model, the function $\Phi(x, y; b)$ satisfies for $0 < x < b$ the equation

$$\frac{\sigma^2}{2}\,\frac{\partial^2\Phi(x, y; b)}{\partial x^2} + \mu\,\frac{\partial\Phi(x, y; b)}{\partial x} - \delta y\,\frac{\partial\Phi(x, y; b)}{\partial y} = 0, \qquad \text{(R4)}$$

which is the same partial differential equation as (4.3) of the paper and can be proved by the same method. We note that (R4) can be obtained from (R3) as a limit, because the Brownian model can be viewed as a limit of the compound Poisson model, as we mentioned in the paper. Let us illustrate this with a family of compound Poisson models of constant jump size. For $\varepsilon > 0$, we consider the compound Poisson model with the jump size distribution function

$$F(z) = F(z; \varepsilon) = \begin{cases} 0 & z < \varepsilon \\ 1 & z \geq \varepsilon \end{cases}$$

and parameters $c = c(\varepsilon)$ and $\lambda = \lambda(\varepsilon)$ such that the first two moments are matched,

$$\mu = c - \lambda\varepsilon, \quad \sigma^2 = \lambda\varepsilon^2. \qquad \text{(R5)}$$

Then, in the limit $\varepsilon \to 0$, the Brownian motion model with parameters μ and σ^2 is obtained. What happens to (R3) in the limit? We may assume $\varepsilon < x$. Then, $F(x) = 1$, and the right-hand side of (R3) is zero. Also,

$$\int_0^x \Phi(x - z, y; b)\, dF(z) = \Phi(x - \varepsilon, y; b),$$

which, by the Taylor series expansion, is

$$\Phi(x, y; b) - \varepsilon\,\frac{\partial\Phi(x, y; b)}{\partial x} + \frac{\varepsilon^2}{2}\,\frac{\partial^2\Phi(x, y; b)}{\partial x^2} + \cdots$$

Using this and (R5), we see that, in the limit $\varepsilon \to 0$, equation (R3), indeed, becomes the partial differential equation (R4).

Let $\{S(t)\}$ be a compound Poisson process independent of the standard Wiener process $\{W(t)\}$ and with Poisson parameter λ and positive summands whose common probability distribution function is F. Suppose that the unmodified surplus process is modeled as

$$X(t) = x + ct + \sigma W(t) - S(t); \qquad \text{(R6)}$$

then the function $\Phi(x, y; b)$ satisfies for $0 < x < b$ the equation

$$\frac{\sigma^2}{2}\,\frac{\partial^2\Phi(x, y; b)}{\partial x^2} + c\,\frac{\partial\Phi(x, y; b)}{\partial x} - \delta y\,\frac{\partial\Phi(x, y; b)}{\partial y} - \lambda\Phi(x, y; b) + \lambda\int_0^x \Phi(x - z, y; b)\, dF(z) = -\lambda w(0)[1 - F(x)], \qquad \text{(R7)}$$

which contains both (R3) and (R4) as special cases. We observe that (R7) can be reformulated in terms of the *infinitesimal generator* of the process $\{X(t)\}$.

Albrecher's second and final comment points out an important distinction. In our paper, we consider the expected utility of the sum of the discounted dividends, while other authors study a related, but substantially different problem, which is to maximize the expectation of the sum of the discounted utilities of the dividends. Duffie (2001, chap. 9) calls this *Merton's problem*, in honor of the pioneering work by the Nobel laureate Robert Merton (1969, 1971). Merton's papers are reprinted in Merton (1990). Also see Björk (1998, chap. 14). The problem considered by Hubalek and Schachermayer (2004), mentioned by Albrecher, is difficult in that, in their model, as in ours, the dividends stop at the time of ruin.

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"Credit Standing and the Fair Value of Liabilities: A Critique," Philip E. Heckman, January 2004

M. W. CHAMBERS*

Dr. Heckman deserves high praise, even accolades, for developing this bold illumination of the folly of reflecting the credit standing of the obligated entity in the calculation of the fair value of its liabilities for presentation in its public accounts. He has managed to cut through the advocates' circular arguments (which are grounded in misguided theory) to expose their specious foundations.

In his introduction, Dr. Heckman rightly decries the failure of the authors of the American Academy of Actuaries *Public Policy Monograph on Fair Valuation of Insurance Liabilities* to take a position on the credit standing issue. Let me say, as a member of the task force that developed the monograph, that he is not the only one who is dissatisfied with the document in that respect. There were at least three of us in the group who felt that the monograph should specifically reject reflection of credit standing. At the same time, there were at least three strong advocates of the opposite view. The latter were of no mind to have their view suppressed, so ultimately, if the task force was to produce any document at all, the two extreme positions were forced to take the middle road of noncommittal. Unfortunately, the monograph is a lesser document as a result.

Dr. Heckman ties the problem to two particular roots. The first root is the ingrained thinking of the accounting profession which has been shaped by more than a half century of commitment to the 'religion' of historical cost accounting. The long history of that approach to the preparation of public accounts has deluded many of our accounting brethren into believing that certain of the constraints that defined that specific methodology are axiomatic truths that apply in all conceivable accounting regimes. The primary example of this, as pointed out by Dr. Heckman, is FASB's assertion in 2000 that "the act of borrowing money at prevailing interest rates should not give rise to either a gain or a loss." Why? Why not? Certainly FASB has not provided a logical, meaningful response. Dr. Heckman has, in this paper, demonstrated that this presumption on FASB's part is mistaken.

Unfortunately, this myth is not confined to FASB. Latterly, in its plans for Phase I implementation of International Accounting Standards for insurance contracts, the IASB continues to cling to this misconception by declaring that there should be no profit (or loss) arising in financial statements at the point of sale of an insurance contract.

The second basic root of the problem is that the accountants have turned to financial economics to develop their views of fair value. Now, financial economists have developed some wonderful techniques for assessing and valuing risk. Unfortunately,

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