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**BRUNO DE FINETTI, THE ACTUARIAL SCIENCES
AND
THE THEORY OF FINANCE IN THE XX CENTURY**

1. Bruno de Finetti is universally known as a lofty mathematician, superlative probabilist and refined scholar of actuarial sciences; but only few in Italy and less than ever abroad (except for some friends of the Italian actuarial academic circles as e.g. P.Boyle, H.Buhlmann and H.Gerber), are aware of the big relevance of the contributions he gave to the foundations of the modern theory of finance, a non elective field for him. The goal of this note is both to shed light on these treasurer-trove contributions, and to reflect on the motivations explaining why they remained so long neglected and unknown.

2. To understand the link connecting B. de Finetti to economics and finance you should keep in mind three names: **Ulisse Gobbi**, **Vilfredo Pareto** and the insurance company **Assicurazioni Generali**. As a young student of the faculty of Sciences in Milano, de Finetti attended a free course in Insurance Economics (really a course in economics of uncertainty), by Ulisse Gobbi, which left an enduring trace in his mind¹. Quite likely induced also by this cultural incentive, de Finetti accepted in 1931 (25 years old mathematician with a reputation of innovative probabilist) a job proposal coming from the Generali Insurance of Trieste. There he found the opportunity to face with concrete insurance problems, while at the same time keeping in touch with the world of actuarial sciences, whose national and international meetings he regularly attended since the beginning of the thirties. Finally, to the work and thought of Vilfredo Pareto he resorted in the thirties to find a reliable methodological background to make head or tail in the theoretical and applied economics of those turmoil years. While not fully accepting the Paretian system, de Finetti treated as pinpoint to any approach to pure and applied economics the Paretian conceits of **ofelimità** (ordinal utility) and **optimum** (the set of allocations which, under a plurality of evaluation criteria, may be changed only worsening at least with respect to one criterion). De Finetti's work "**Il problema dei pieni**", presented in 1938 to a competition announced by the Italian National Research Council, and later (1940) published as winner on the G.I.I.A.(8), turns out as the result of these human events and cultural propensities. We think it is surely one of the more relevant writings in the history of modern finance: there new ideas and methods are introduced and discussed (even if not always exhaustively and coherently), largely earlier than other authors, today universally credited with those ideas and methods. It could be safely said that, at least in the field of theoretical finance, de Finetti sowed but was not able to mow.

3. As said before, in 1938 the National Research Council announces a competition for the best work on the subject: "On the maximum amount which an insurance company may accept as its own retained risk. Theoretical contributions with reference to the real insurance world, keeping account of reinsurance opportunities", and de Finetti is awarded the first prize. In his statement the problem is seen as a proportional reinsurance one, with decisional variables the retention quotas of the company's portfolio. Under proportional reinsurance, argues de Finetti, each additional reinsurance has a twofold effect. It lowers the risk of the

retained portfolio but at the same time lowers its profitability. Moreover risk and profitability may well be captured respectively by the variance (a quadratic function of the retention quotas) and by the expectation (a linear function) of the retained portfolio. And coherently with his economic ideas, this looks as a typical (two criteria, mean-variance indeed) optimum problem, contrarily to the approach prevailing then in actuarial circles, exclusively concerned with the control of risk. This is to be seen as the original proposal to apply the **mean-variance** approach to face portfolio problems under uncertainty. And this is by no means only a methodological innovation. Looking for a system to solve concrete reinsurance problems and making recourse, as usual for him, to brilliant geometrical constructions, he offers a procedure to obtain the optimum set, in the n dimensional space of retention quotas, as a sequence of line segments, joining the vertex **(1)** of the unit hypercube corresponding to full retention of all policies with the vertex **(0)** of total reinsurance. On this “optimum reinsurance path” the extreme of the segments are the points corresponding to the entrance in active reinsurance of another policy, joining some other already partially reinsured. Later we will come back to this point. A digression is in order now: indeed de Finetti’s proposal comes about a dozen years before the issuing by H. **Markowitz**, of the papers (13), (14) (based on the same mean-variance approach, although concerning a financial rather than a (re)insurance portfolio), which will bring him the 1990 Nobel Price in Economics and the universal recognition as the **founder of modern finance**. On the contrary, de Finetti’s contribution stayed unknown to the world of economics and finance and confined to a restricted actuarial circle, unable to appreciate its relevance as a forerunner of the portfolio selection approach. A concurrence of circumstances may explain such negligence. The segmentation prevailing at that time, at least in continental Europe, between scholars of economics and actuarial sciences; then the language barriers, keeping account that de Finetti’s paper was published in Italian language, on an Italian review with international reputation only in the actuarial world. Things were made even worse by the coincidence of the publication with the beginning of the second world war and the consequent breakdown of at least some relevant international contacts. Last but not least we ought to underline that the author did not ever claim his primacy or even stress his contribution. Quite likely he himself did not realize the relevance of his work; indeed he did not insert it in the (long) list of his papers linked (even weakly) to economic problems². If a self quotation is allowed here, the first pointing out of de Finetti’s primacy, along with a (non exhaustive) comparison of the approaches of the two authors, is contained in a paper by myself (19) appeared in 1985. But now I am not proud of that; rather I regret having been too shy and having chosen (once more) a not appropriate vehicle (proceedings of an actuarial meeting). Then we had to wait twenty years more, through a more incisive interaction with the English speaking financial world of some younger Italian scholars (Claudio Albanese, Luca Barone and Francesco Corielli), to have a top level international recognition. It came in the words of M. Rubinstein in (21): “*it has recently come to the attention of economists in the English speaking world that among de Finetti’s papers is a treasurer-trove of results in economics and finance written well before the work of the scholars that are traditionally credited with these ideas.....de Finetti’s 1940 paper anticipating much of mean variance portfolio theory later developed by H.Markowitz*”, and of Markowitz himself in (15):” *it has come to my attention that, in the context of choosing optimum reinsurance levels, de Finetti essentially proposed mean variance portfolio analysis using correlated risks*”. After that let us go back to the computational side of the matter. Starting from the no correlation case, de Finetti suggests that a key role in the operational procedure to find the optimum set is played by the functions

$$F_i(\underline{x}) = \frac{1}{2} \cdot \frac{\partial V / \partial x_i}{\partial E / \partial x_i} = \sum_j x_j \cdot \frac{V_{ij}}{m_i}$$

Denoting by \mathbf{x} the vector of retention quotas and by V and E variance and expectation of the retained portfolio, by V_{ij} the covariance between a couple (i,j) of policies and by m_i the expected gain of policy i , the above functions represent a synthetic indicator (in expected gain numeraire) of the advantage (lowering of variance) coming from a small local additional reinsurance of the i -th policy. Accordingly and coherently with the goal of maximum lowering of variance in expected gain numeraire, they drive the determination of the entrance ordering (in reinsurance) of the policies as well as the direction of the optimum segments in the n dimensional space of the retentions (for any given group of policies jointly reinsured). It is interesting to give here a quick characterization of the fundamental result obtained by de Finetti for the no correlation case and incautiously (as we shall see) extended to the general case with correlated risks. Let $M = \max F_i(\mathbf{1})$, and denote by $I(\mathbf{x})$ e $I_1(\mathbf{x})$ the sets defined so as $i \in I$ iff $0 < x_i < 1$, respectively $i \in I_1$ iff $x_i = 1$. Then for any given real F between 0 and M there is a unique optimum $\mathbf{x}(F)$; for $F=0$ it is $\mathbf{x}(0) = \mathbf{0}$, whereas for F positive, $\mathbf{x}(F)$ satisfies the following conditions: $F - F_i(\mathbf{x}) = 0$ if $i \in I$, $F - F_i(\underline{x}) \geq 0$ if $i \in I_1$. If the equality holds for at least one $i \in I_1$ then \mathbf{x} is extreme point of one of the segments of the optimum path. In particular if I is void the vertex $\mathbf{1}$. Note that this means there are no optimum points with some zero elements, except of course for the endpoint $\mathbf{0}$ of the optimum path. This result holds unconditionally only in the no correlation case and his extension to the general case was a surprising error of de Finetti. Yet more surprising is that nobody discovered it until Markowitz himself a few months ago³. Quite likely de Finetti was biased by the correct thought that the proposition should hold for sure with parameters (mean vector and covariance matrix) reflecting realistic market values of the safety loading and correlation coefficients. For some details on this critical point see (20). Anyway let us write what should be the correct version of de Finetti's result. Denote by I , I_1 , I_0 the sets defined so as $i \in I$ iff $0 < x_i < 1$, $i \in I_1$ iff $x_i = 1$, respectively $i \in I_0$ iff $x_i = 0$. Then for any given real F between 0 and M there is a unique optimum $\mathbf{x}(F)$ (note that this does not exclude that for some interval of F values where I is void \mathbf{x} is the same for any F in the interval); for $F=0$ it is $\mathbf{x}(0) = \mathbf{0}$, for F positive, $\mathbf{x}(F)$ satisfies the following conditions: $F - F_i(\mathbf{x}) = 0$ if $i \in I$, $F - F_i(\underline{x}) \geq 0$ if $i \in I_1$, $F - F_i(\underline{x}) \leq 0$ if $i \in I_0$. It is interesting to note that, as recognized by Markowitz (2005), along this road de Finetti was going very close to the so called **global optimality conditions**⁴ in quadratic programming, later introduced by Kuhn-Tucker (12) (and previously discovered by Karush), which gave Markowitz a powerful tool to support his "**critical line algorithm**"⁵. Roughly resuming, a proper adjustment of de Finetti's optimum line procedure could be seen (for the correlation case too) as an early version of the critical line algorithm.

4. This would be enough to qualify "Il problema dei pieni" as a relevant paper in the history of the theory of finance. But, as we shall see, in the second part of the paper there are other very interesting ideas. After having developed in a first stage and in a single period setting the mean variance optimum set, de Finetti faces the problem to select in a second stage a single point from the optimum. And to reach the goal moves to a multiperiod horizon, indeed to an asymptotic one, aiming to choose a strategy consistent with a given acceptable (asymptotic) ruin probability. The background here comes from the casting of schemes of risk theory (as developed mainly by the actuarial skandinavian school⁶) with the well known probability models named **gambler's ruin**. Such models consider an infinite sequence of fair games (expectation zero, conditionally to any past path of the game) played by two agents, and show that the asymptotic ruin of each player is the ratio between the initial wealth of the competitor and the overall initial wealth of both players. Hence in the asymmetric case (only one player endowed with unlimited wealth) the ruin of the weak agent is sure. De Finetti mimics the insurance company situation as the one of a gambler

with finite wealth facing an asymmetric game, but whose ruin is not sure owing to safety loadings, which modify the game from fair to advantageous. In this scenario de Finetti obtains the following fundamental result: a company with initial wealth G , which follows a strategy to retain a sequence $X_1, X_2, \dots, X_h, \dots$ of single period independent portfolios, characterized by a common coefficient β satisfying $E(\exp(-\beta X_h)) = 1$ for each $h=1, 2, \dots$ has asymptotic ruin probability $p = \exp(-G\beta)$. Then any choice of the couple (G^*, p^*) , initial wealth and ruin probability, uniquely defines a value of the common coefficient $\beta^* = (-1/G^*) \ln(p^*)$ and hence a unique sequence of retained portfolios belonging to the respective optimum and satisfying $E(\exp(-\beta X_h)) = 1$.

5. Summing up, this line of reasoning allows de Finetti to show that the asymptotic ruin probability of an insurance company, endowed with initial wealth G and which follows reinsurance strategies such that its retained portfolio is characterized, for each period, by a coefficient β , equals $\exp(-\beta G)$. According to this approach the issue is mathematically clear, but rather obscure as to the economic meaning. Only after a careful reflection, it may be realized that organizing a sequence of portfolios, characterized by a common coefficient β , is equivalent to accept a **sequence of indifferent games under exponential utility** with coefficient β . More formally describes the behaviour of a company whose utility of money is $u(x) = 1 - \exp(-\beta x)$, which in any period retains a portfolio tailored so as its **expected utility does not change**. Thus it could be said that, in the second part of the paper, de Finetti was really sketching, although in the particular exponential case, the **expected utility paradigm**, another anticipation of a pillar of the foundations of the modern finance.

6. But at that time de Finetti was not aware of the importance of his suggestions. He clearly perceived it only some years later, after reading the fundamental work (22) by Von Neumann-Morgenstern, where a neo-bernoullian theory of measurable (up to linear transforms) utility, linked to preferences regarding random variables, was coherently exposed⁷. And recognizing the connections between his approach and the new paradigm, was able to define in another path breaking paper (9) some key concepts of the utility theory. In detail and with the aim to define proper measures of risk aversion associated to a given cardinal utility, he introduced the **absolute risk aversion function** $(-u''/u')$, invariant to linear transforms of the utility function u ; the **probability premium** (the difference between winning and losing probability which makes indifferent a bet of amount h); the **risk premium** (the sure loss indifferent to a fair bet of amount h). Then proved that such premiums are (at least for "small values" of h) directly proportional to the value (at the starting wealth) of the risk aversion function. Moreover the exponential utility, $u(x) = 1 - \exp(-\alpha x)$, was recognized as the one associated to an attitude of **risk aversion** at the **constant level** α . And such attitude was indeed linked, although in a short remark, to the asymptotic theory of risk, with the explicit assertion that "*the classical criterion of the riskness level (i.e. the one exposed in "Il problema dei pieni"), is coincident with the utility criterion (be careful in the indifference sense and not in the optimizing one) under constant risk aversion*". These are results of big relevance in the foundations of economics and, until a few months ago, universally credited to papers by Arrow (1) and Pratt (18), written a dozen years later than the path breaking work of de Finetti. Also the revindication of this primacy came from the Trieste actuarial school in the eighties⁸ (see (5) and (6)) and found international imprimatur once more by Rubinstein in (21): "*in 1952 anticipating K.Arrow and J.Pratt by over a decade, he formulated the notion of absolute risk aversion, used it in*

connection with risk premia for small bets and discussed the special case of constant risk aversion”.

7. The last reference regards the link between de Finetti's ideas and another pillar of the theory of modern finance: the arbitrage free pricing principle, applied by Black-Scholes (2), Merton (16) and Cox-Ross-Rubinstein (4) to evaluate options and other derivatives and then rigorously formalized by Harrison-Kreps (11). Indeed this connection is of an entirely different type, being based on a sort of methodological inversion. Concerning this point let us recall that de Finetti, besides being a mathematician able to apply mathematical methods and models to economics, finance and actuarial sciences, was also prone to use economic concepts to work in the elective fields of mathematics and probability. And the more striking example of this outlook is embedded in the definition of probability of an event as the price of an asset with random return linked to the logic value (true or false) taken by the event. An economic reasoning about the conditions that prices of some logically connected investments should obey, took de Finetti to define a proper coherency condition. While from a mathematical point of view this condition lies at the basis of the subjective approach to probability theory, in economic and financial applications it is the exact counterpart of the arbitrage free pricing principle. For some details see (7). And it is worth noting that this condition applies both to single period problems (theorem of total probability) and, upon introduction of the concept of conditional probability (defined economically as the price of an event conditional to some information), also to multiperiod problems (theorem of compound probability). Truly an exciting modern approach!

8. At the end of this excursus we may conclude that De Finetti was an extraordinary forerunner of paradigms (mean variance, expected utility, arbitrage free pricing), central both in theory as well as in real world (investment funds, options and other derivatives) modern finance. A positive role in this picture was surely played by the experience in the insurance sector and the connected strong contact with actuarial sciences. On the contrary we must register the lack of an organic link with the university structures (which he was able to obtain only at the end of the thirties but with concrete effects only about ten years later). Quite likely he was thus deprived of scholars and academic interlocutors, able to a timely dialogue with him, so as to avoid some inaccuracies (like the wrong last segment conjecture) and above all to provide more easy contacts with the English speaking world in economics and finance. Indeed he could be able to reach these links only in the fifties and in the sixties; moreover they were then addressed mainly toward probability (Savage) and econometric (Frish, Morishima, Tinbergen) applications. This at least partially explains why only now in the centenary of his birthday, we may joyfully say that this gap has been (or better is going to be) finally filled.

FOOTNOTES

¹According to his own words in [10] pag. 26 footnote 3: *“I appreciated very much and it left me with an enduring memory of lessons opening to me new horizons”*.

²”Il problema dei pieni” is not included in the list of 46 papers connected to economics, given by de Finetti at the end of [10], pag. 335!

³ *“de Finetti last segment conjecture is not correct”* in [15].

⁴ It could be safely said that the Finetti applied a “one side version” of the global optimality conditions.

⁵ This being the case the assertion by Markowitz, in [15], that *“de Finetti did not solve the problem of computing mean variance efficient reinsurance frontier with correlated risks”*, although formally correct, sounds rather cavalier. Some remarks about the possibility to fit de Finetti’s procedure to the correlation case, in a forthcoming paper by F.Pressacco and P.Serafini (full professor of operations research at the University of Udine).

⁶ Relevant contributions to the theory of risk have been given also by the Italian actuarial school. Besides de Finetti see the papers by Cantelli [3] e Ottaviani [17].

⁷ It should be remarked that the Finetti expressed some time later his regret not having being able to properly and timely apply the expected utility approach. Indeed he commented in [10] pag. 69: *“this way to introduce and define expected utility was very close to the one proposed by myself (in 1930). The difference was that I intended to base on these ideas only the concept of probability, without caring utility. The source of my reluctance came from motivations that now I am recognizing as groundless....I looked upon the idea of Pareto to give up measurable utility as a valuable progress of the scientific thinking, and I did not like to take a step backward that point.....Hence a self critical attitude (ibidem pag. 67), not for a personal concern, but rather as warning about the difficulties to avoid unconscious mental obstructions, coming even from those fighting against them”*.

⁸ Starting from the sixties the favourite de Finetti’s scholars, L.Daboni. C. de Ferra and D.Furst widely and successfully applied the expected utility paradigm to theory and practice of insurance.

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