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De Finetti’s contribution to the theory of random functions

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The name of Bruno de Finetti is indissolubly linked with the subjectivistic conception of probability.

However, de Finetti achieved fame also for crucial contributions both to other branches of probability and to important fields of mathematics, to say nothing of the philosophic debate at large.

Limiting myself to the field of probability, I would like to focus on his ground-breaking research on random functions and, in particular, on *random functions* with *stationary and independent increments*, in connection with the birth of the *subjectivistic conception of probability*.

Hence, a more appropriate title of this talk could be

The dawn of the theory of random functions and of the subjectivistic approach to probability.

A tangled story of results and thought about probability.

Contents

5. Further contributions (continuity of trajectories and continuity of the instantaneous law, integration of random functions, . . .).

6. Birth and consolidation of the subjectivistic conception: weak **versus** strong version of the principle of total probabilities (additivity); ensuing criticism of the methods used to study random functions.

. . . The essential novelty in the scientific method would then be the substitution of the logic by probability theory; instead of rationalistic science, where certainty is deduced from certainty, there would be a probabilistic science, where the probable is deduced from the probable. It is not prejudicially necessary to renounce determinism for setting up science on these bases; we may confess not to be able to foresee an event without saying that forecasting is itself impossible . . .

(de Finetti: Le leggi differenziabili e la rinunzia al determinismo, 1930)

Rigid laws which state that a certain fact is bound to occur in a certain way are being replaced by statistical laws stating that a certain fact can occur depending on a variety of ways governed by probability laws.

(see, also, Kolmogorov: ″Über die analytischen Methoden
in der Wahrscheinlinchkeitsrechnung″, 1931)

″Le Leggi differenziali . . . ″ reproduces a talk given at *Seminario matematico della Facoltà di Scienze dell'Università di Roma* on 5th April 1930 about

Random Functions

after the publication of 3 ″Note lincee″

- Sulle funzioni a incremento aleatorio (1929)
- Sulla possibilità di valori eccezionali per una legge di incrementi aleatori (1929)
- Integrazione delle funzioni a incremento aleatorio (1929)

In $''$Sulle funzioni a incremento aleatorio $''$ (1929) de Finetti starts out from a probabilistic version of Volterra's classification of ordinary laws of physics (blue for classical, red for probabilistic):

(a) $X_t' = f(t)$

$\mathcal{L}(X_t - X_{t_0} | X_u,\, 0 \leq u \leq t_0) = \mathcal{L}(X_t - X_{t_0})$ (known law)

(b) $X_t' = f(t, X_t)$

$\mathcal{L}(X_t - X_{t_0} | X_u,\, 0 \leq u \leq t_0) = \mathcal{L}(X_t - X_{t_0} | X_{t_0})$ (differential law)

(c) $X_t' = f(t; X_u,\, 0 \leq u \leq t)$

$\mathcal{L}(X_t - X_{t_0} | X_u,\, 0 \leq u \leq t_0)$ (integral law)

The result is a classification for stochastic processes.

″Funzioni a incremento aleatorio con legge nota″ stands for ″random functions with random increments having a known law″, i.e.

Stochastic processes with independent increments

(Case (a) of the previous classification).

de Finetti limits himself to considering this case.

$0 \leq t_0 < t_1$ $\qquad X_{t_1} - X_{t_0}$

$F_{t_0,t_1} \longleftrightarrow \psi_{t_0,t_1}$

$\psi_{0,t+\Delta} = \psi_{0,t} \cdot \psi_{t,t+\Delta}$

$$\frac{1}{\Delta}\left(Log\psi_{0,t+\Delta} - Log\psi_{0,t}\right) = \frac{1}{\Delta}Log\psi_{t,t+\Delta}$$

$$\frac{\partial}{\partial t}Log\psi_{0,t} = \lim_{\Delta\to 0}\frac{1}{\Delta}Log\psi_{t,t+\Delta} =: Log(\psi_t^*)$$

ψ_t^* is what de Finetti suggested should be called (characteristic function of the) derived law

He defined a $''$known$''$ law for the increment to be *fixed* when it is independent of t:

$$\psi_t^* = \psi_1^* \qquad \text{for every } t > 0.$$

(*Process with independent and stationary increments*). Then

$$t Log \psi_1^* = \int_0^t \frac{\partial}{\partial s} Log \psi_s ds = Log \psi_t$$

$$\psi_1^*(\xi) = (\psi_t(\xi))^{1/t} \qquad (\text{and } \psi_1^* = \psi_1)$$

Then, (de Finetti, 1929), the instantaneous law of a process with stationary and independent increments must be infinitely divisible (term introduced by Khintchine in 1937).

So, if X_1 has finite second moment, i.e.

$$Log\psi_1(\xi) = im_1\xi - \frac{\xi^2}{2}\sigma_1^2 + o(\xi^2) \qquad (\xi \to 0),$$

from $\psi_t = \psi_1^t$ one gets

$$m_t := E(X_t) = tm_1, \quad \sigma_t^2 := Var(X_t) = t\sigma_1^2.$$

De Finetti stresses the importance of the case in which

$$Log\psi_1(\xi) = im_1\xi - \frac{\xi^2}{2}\sigma_1^2$$

i.e. the case of increments with *known and fixed Gaussian law*.

De Finetti concludes ″Funzioni a incremento aleatorio″ (1929) with the fundamental statement

Almost every path of a random function with known and fixed Gaussian law has nowhere finite lower and upper derivatives

All literature refers to this statement as a Paley, Wiener and Zygmund (*) theorem, ignoring de Finetti's authorship.

(*) Mathematische Zeitschrift (1933)

Everything seems to point to the fact that de Finetti was actually unaware of fundamental contributions to random functions such as:

Bachelier’s *Théorie de la spéculation* (1900)

Wiener’s *Differential–space* (1923)

Daniell’s *Integral product and probability* (1921)

Bachelier had discovered a few distinguished properties of the standard Brownian motion process like

$$Prob\{\sup_{0 \le t \le T} X_t \ge \lambda\} = 2Prob\{X_T \ge \lambda\}.$$

The Wiener (1923) concept of *differential-space* brings Wiener's and de Finetti's stances pretty close to each other.

In fact, Wiener took his cue from the physical theory of the Brownian movement, which led him to state that is the displacement of a particle over an interval that is *independent* of the particle over another interval. (The same as de Finetti's concept of ″known″ law.)

$''$That is, instead of $f(1/n), \ldots, f(k/n), \ldots, f(1)$ representing 'dimensions' of $f(t)$, the n quantities

$$\begin{aligned} x_1 &= f(1/n) - f(0) \\ x_2 &= f(2/n) - f(1/n) \\ &\ldots\ldots \\ x_n &= f(1) - f((n-1)/n) \end{aligned}$$

are of equal weight, vary independently, and in some degree represent dimensions.$''$

Firstly, Wiener tries to justify the assumption that the prob. distr. of $f(t_1) - f(t_0)$ is Gaussian $(0, A(t_1 - t_0))$, for any t_0, t_1, $0 \leq t_0 < t_1 \leq 1$.

His goal was to define mean values for certain functionals on the set of all real–valued functions defined on $[0, 1]$.

To obtain a Daniell integral that gives these functionals, he started out from functionals that depend only on the path's values at a finite number of time points. He found their mean value using the above Gaussian distributions.

By the Daniell extension method, he was able to define mean values for a wide class of functionals. In particular he obtained mean values for indicators of events such as

$$\{f \in C[0,1]\} \qquad \{f \in BV[0,1]\}$$

and he established that

$$Prob\{f \in C[0,1]\} = 1 \qquad Prob\{f \in BV[0,1]\} = 0.$$

(de Finetti often assumes continuity of trajectories, but he never deals with the problem of deducing that property from ″distributional″ information.)

The second paper, in order of importance, de Finetti wrote about processes with stationary and independent increments is

$''$Le funzioni caratteristiche di legge istantanea$''$ (1930)

where he resolves on characterizing ψ_1 (the $''$derived law$''$). Clearly,

$$Log\psi_1 = \lim_{n\to+\infty} n\{e^{\frac{1}{n}Log\psi_1} - 1\} = \lim_{n\to+\infty} n\{e^{Log\psi_{1/n}} - 1\}$$

or, more in general,

$$Log\psi_1 = \lim_{\Delta\to 0} \frac{1}{\Delta}\{e^{Log\psi_\Delta} - 1\}$$

$$\psi_1 = \lim_{\Delta\to 0} \exp\{\frac{1}{\Delta}(\psi_\Delta - 1)\}.$$

Now,

$$\frac{1}{\Delta}(\psi_\Delta(\xi) - 1) = \int_{\mathbb{R}} \frac{1}{\Delta}(e^{i\xi x} - 1)dF_\Delta(x) = \int_{\mathbb{R}} (e^{i\xi x} - 1)\mu_\Delta(dx)$$

where

$\mu_\Delta(A) =$ expected number of increments, on intervals of length Δ, belonging to A.

One can set up approximating sums of the integral such that

$$\int_{\mathbb{R}} (e^{i\xi x} - 1)\mu_\Delta(dx) = \sum_j \lambda_j^{(\Delta)}(e^{i\xi x_j^{(\Delta)}} - 1)$$

and, thus,

$$\psi_1(\xi) = \lim_{\Delta \to 0} e^{\sum_j \lambda_j^{(\Delta)} (e^{i\xi x_j^{(\Delta)}} - 1)}$$

$$= \lim_{\Delta \to 0} \prod_j e^{\lambda_j^{(\Delta)} (e^{i\xi x_j^{(\Delta)}} - 1)}$$

that is

The derived law (any infinitely divisible law) coincides with the limit of a finite convolution of Poisson type.

Kolmogorov (*) starts out from de Finetti’s derived law,

$$\int_{\mathbb{R}} \frac{1}{\Delta}(e^{i\xi x} - 1)dF_{\Delta}(x),$$

under the assumption that

$$\int_{\mathbb{R}} x^2 dF_{\Delta}(x) < +\infty \qquad (\forall \Delta > 0).$$

(*) *Sulla forma generale di un processo stocastico omogeneo. (Un problema di Bruno de Finetti), Rend. Lincei, 1932.*

He writes

$$\frac{1}{\Delta}(\psi_\Delta(\xi) - 1) = i\xi m + \frac{1}{\Delta}\int_{\mathbb{R}} (e^{i\xi x} - 1 - i\xi x) dF_\Delta(x)$$
$$= i\xi m + \int_{\mathbb{R}} p(x,\xi) dG_\Delta(x)$$

with

$$G_\Delta(x) = \frac{1}{\Delta}\int_{-\infty}^{x} u^2 dF_\Delta(u)$$

and

$$p(x,\xi) = \begin{cases} \frac{e^{i\xi x} - 1 - i\xi x}{x^2} & x \neq 0 \\ \frac{-\xi^2}{2} & x = 0. \end{cases}$$

Then, Kolmogorov proves that $\int_{\mathbb{R}} p(x,\xi)dG_{\triangle_n}(x)$ converges to $\int_{\mathbb{R}} p(x,\xi)dG(x)$ along some subsequence $(\triangle_n)_{n\geq 1}$, to obtain

$$\psi_1(\xi) = \int_{\mathbb{R}} p(x,\xi)dG(x)$$

with

$$p(x,\xi) = \begin{cases} \frac{e^{i\xi x}-1-i\xi x}{x^2} & x \neq 0 \\ \frac{-\xi^2}{2} & x = 0. \end{cases}$$

In 1937, Khintchine (*) used the de Finetti-Kolmogorov method to derive the Lévy general representation of the characterisic function of an infinitely divisible law.

(*) *A new derivation of a formula by P.Lévy*, Bulletin of the Moscow State University.

From

$$Log\psi_1(\xi) = \lim_{\Delta\to 0}\int_{\mathbb{R}} \frac{1}{\Delta}(e^{i\xi x}-1)dF_{\Delta}(x),$$

setting

$$K_{\Delta}(x) = \int_0^x \frac{u^2}{1+u^2}\frac{1}{\Delta}dF_{\Delta}(u),$$

he obtains

$$Log\psi_1(\xi) = \lim_{\Delta\to 0} \int_{\mathbb{R}} (e^{i\xi x} - 1) \frac{1+x^2}{x^2} dK_\Delta(x)$$

$$= \lim_{\Delta\to 0} [i\xi\gamma_\Delta + \int_{\mathbb{R}} (e^{i\xi x} - 1 - \frac{i\xi x}{1+x^2}) \frac{1+x^2}{x^2} dK_\Delta(x)]$$

(with

$$\gamma_\Delta = \int_{\mathbb{R}} \frac{1}{x} dK_\Delta(x))$$

and concludes that

$$Log\psi_1(\xi) = i\xi\gamma + \int_{\mathbb{R}} (e^{i\xi x} - 1 - \frac{i\xi x}{1+x^2}) \frac{1+x^2}{x^2} dK(x).$$

Lévy played a fundamental role in the story of processes with independent increments, not only because he was the first to deduce a representation for general infinitely divisible distributions in a paper of 1934 (in *Annali della Scuola Normale Superiore di Pisa*), but also because he started out with an in–depth analysis of the properties of the paths of a process with stationary and independent increments.

In point of fact, he derived the above general representation from that analysis.

In other words, his approach is quite different from de Finetti's and Kolmogorov's approach. Lévy recalls that he became aware of de Finetti's and Kolmogorov's papers after writing and submitting his own paper.

De Finetti's remaining papers about the subject of processes with stationary and independent increments contain:

- A proof of the fact that the law of X_t must be continuous when the trajectories of $t \mapsto X_t$ are continuous. (*Sulle possibilità di valori eccezionali...* 1929; *Le funzioni caratteristiche di legge istantanea dotate di valori eccezionali*, 1931)

- A discussion of the law of $\int_0^t X_u du$ when X_t has continuous trajectories (*Integrazione delle funzioni a incremento aleatorio*, 1929).

Although they are not of the same level as those described in the first part of this talk, they are anyway of a certain importance when it comes to understanding the influence exerted by the ultimate presentation of the mathematical theory of subjectivistic probability on de Finetti's critical look at his work in the field of random functions and on its rapid abandonment.

Recall that his subjectivistic stance, based on the coherence principle, does not prescribe σ–additivity. Hence, probabilistic statements, to be general, must involve simply additive probability distributions.

So, as to the problem of integration of $t \mapsto X_t$, one has

$$I_t := \int_0^t X_u du = \lim_{n \to +\infty} \frac{t}{n} \sum_{h=1}^{n} X_{th/n}.$$

Then, setting

$$S_n = \frac{t}{n} \sum_{h=1}^{n} X_{th/n} = tX_0 + \frac{t}{n} \sum_{h=1}^{n} (n-h+1)\{X_{th/n} - X_{t(h-1)/n}\}$$

one obtains

$$Log\phi_{S_n}(\xi) \to ic\xi t + \frac{1}{\xi} \int_0^{t\xi} Log\psi_1(u)du =: Log\widetilde{\phi}(\xi)$$

whenever $X_0 = c$.

However, giving up complete additivity, one cannot conclude that $\tilde{\phi}$ is the characteristic function of integral I_t.

Patrizia Berti and Pietro Rigo have shown that information about the projective system of X_t has no influence in determining the law of I_t.

(An analogous problem appears with respect to the classical law of large numbers for sequences of events. (*Sui passaggi al limite...*, 1930).)

Moreover, in a finitely additive setting, nowhere differentiability of paths of a Brownian motion represents a particular case of the following general proposition:

For any pair of strictly positive numbers ϵ and M, the event that there is some subinterval of $[0, 1]$, with length $> \epsilon$, in which the Lipshitz condition

$$|X_{t_2} - X_{t_1}| \leq M|t_2 - t_1|$$

holds true has probability 0.

To appreciate de Finetti's stance on the role played by complete additivity, with respect to conditions which are necessary for *coherence* of probability assessments, it would be enough to read and ponder over the four short Notes containing the debate with Maurice Fréchet on the value of the postulate of σ–additivity, published in *Rendiconti Ist. Lombardo* (1930).

Nowadays, treatment of probabilistic problems in the framework of non σ–additive probabilities is definitely unusual. Studies of this kind are viewed, in most cases, as mere oddities.

According to de Finetti, though, the matter was of paramount importance. He tried to enlighten the probabilistic community as to the ″relativism″ of statements which, before the appearance of de Finetti's criticism, might have been viewed as descriptions of objective and general truths.

The abandonment of σ–additivity, on the contrary, points out that the value of these ″truths″ could depend on conventional unjustified way of conceiving probability.

As we have seen, de Finetti was, in a sense, a pioneer since he thought it fit to reconsider some of the tenets of his own research, thus blazing exciting new trails. In spite of the difficulties coming large on the horizon he stayed the course as it were, although that spelled isolation within the scientific community. But it is just beacuse of this moral courage of his that he stands out from the rest of the pack.

A remark from ''Le leggi differenziali e la rinunzia...'' on observability

$<<$ The function X_t has a random behaviour. Hence the hypothesis that it satisfies or does not satisy a given condition may result either true or false. Therefore this hypothesis represents an event which might have a certain probability. But in order that, from an empirical point of view, the probability evaluation of an event does not appear to us as meaningless we have to acknowledge at least the theoretical possibility of experimentally verifying when it is true or false; this consideration leads to establish amongst the various conceivable properties of a function X_t a fairly strong distinction $>>$

According to it, only conditions depending on a finite number of values of X_t (= empirical conditions) are susceptible of ″concrete″ probability evaluations.

Semiempirical or transcendental conditions can be considered, provided that they are studied in relation to extra conditions that are extraneous to the essential aspects of a given problem.

In ″Funzioni aleatorie″ (1937), de Finetti assumes a critical attitude toward his previous papers (where he considers the probability of *transcendental conditions*) and, in particular, toward the method of the ″derived″ law.

To ″probabilize″ conditions of an empirical nature it is enough to assess projective systems of finite dimensional distributions.

This is the same view inspiring the Kolmogorov construction of probability laws in infinite–dimensional spaces.

$$0 = t_0 < t_1 < \cdots < t_n \qquad t \to G(t)$$

$$u_k := G(t_k + 0) - G(t_k - 0) \qquad k = 0, \ldots, n$$

$$G(t) = 0 \quad t < t_1$$

$$G(t) = \sum u_k \quad t > t_n$$

$$\phi(G) := E(e^{-i\sum_{k=1}^{n} u_k X_{t_k}})$$
$$= E(e^{-i\sum_{k=1}^{n} G(t_k)\Delta X_k})$$

$$(\Delta_k := X(t_k) - X(t_{k-1}))$$

$$Log\phi(G) = \sum_k \int_{t_{k-1}}^{t_k} Log\psi_t^*(G(t_k))dt$$
$$= \int_0^{+\infty} Log\psi_t^*(G(t))dt.$$