UNCONDITIONAL UNIQUENESS FOR THE CUBIC
GROSS-PITAEVSKII HIERARCHY VIA QUANTUM DE FINETTI

THOMAS CHEN, CHRISTIAN HAINZL, NATAŠA PAVLOVIĆ, AND ROBERT SEIRINGER

Abstract. We present a new, simpler proof of the unconditional uniqueness
of solutions to the cubic Gross-Pitaevskii hierarchy in \( \mathbb{R}^3 \). One of the main
tools in our analysis is the quantum de Finetti theorem. Our uniqueness result
is equivalent to the one established in the celebrated works of Erdős, Schlein
and Yau, [18, 19, 20, 21].

1. Introduction

In this paper, we give a new proof of unconditional uniqueness of solutions to the
cubic Gross-Pitaevskii (GP) hierarchy in \( \mathbb{R}^3 \). The cubic GP hierarchy is a system
of infinitely many coupled linear PDE’s describing a Bose gas of infinitely many
particles, interacting via repulsive two-body delta interactions in the defocusing
case, and via attractive two-body delta interactions in the focusing case. In the
defocusing case, it emerges from the \( N \to \infty \) limit of the BBGKY hierarchy of
marginal density matrices for a bosonic \( N \)-particle Schrödinger system where the
pair interaction potentials tend to a delta distribution as \( N \to \infty \). Factorized
solutions to GP hierarchies are determined by solutions of the corresponding cubic
nonlinear Schrödinger (NLS) equation. In this sense, the NLS is interpreted as
the mean field description of an infinite system of interacting bosons in the Gross-
Pitaevskii limit.

The derivation of the nonlinear Hartree (NLH) equation from an interacting
Bose gas was first given by Hepp in [29]. The BBGKY hierarchy was prominently
used in the works of Lanford for the study of classical mechanical systems in the
infinite particle limit [32, 33]. Subsequently, the first derivation of the NLH via
the BBGKY hierarchy was given by Spohn in [42]. More recently, this topic was
revisited by Fröhlich, Tsai and Yau in [26], and in the last few years, Erdős, Schlein
and Yau have further developed the BBGKY hierarchy approach to the derivation
of the NLH and NLS in their landmark works [18, 19, 20, 21], which initiated much
of the current widespread interest in this research topic. The proof strategy can be
briefly summarized as follows. We consider \( N \) bosons in \( \mathbb{R}^3 \) described by the wave
function \( \Phi_N \in L^2 (\mathbb{R}^3)^N \), where \( \Phi_N (x_1, \ldots, x_N) \) is symmetric under permutation
of particle variables, and satisfies the Schrödinger equation

\[
ix_t \Phi_N = H_N \Phi_N
\]  

(1.1)
with $N$-body Hamiltonian

$$H_N = \sum_{j=1}^{N}(-\Delta x_j) + \frac{1}{N} \sum_{1 \leq i < j \leq N} V_N(x_i - x_j).$$  \hfill (1.2)$$

The pair interaction potential has the form $V_N(x) = N^{3\beta}V(N^\beta x)$ with $\beta \in (0,1]$. $V$ is a sufficiently regular, rotationally symmetric pair interaction potential (we will not specify $V$ in detail since the derivation of the NLS is not the topic of this paper). We assume that $\int V(x)dx = 1$ if $\beta < 1$, and that the scattering length corresponding to $V$ has value 1 if $\beta = 1$. Clearly, $V_N \rightarrow (\int V dx)\delta$ as $N \rightarrow \infty$.

For $k = 1, \ldots, N-1$, the $k$-particle marginal density matrices are obtained from

$$\gamma_N^{(k)} := \text{Tr}_{k+1,k+2,\ldots,N} \langle \Phi_N | \Phi_N \rangle,$$

where $\text{Tr}_{k+1,k+2,\ldots,N}$ denotes the partial trace with respect to the particle variables indexed by $k + 1, k + 2, \ldots, N$. It follows immediately that the property of admissibility holds,

$$\gamma_N^{(k)} = \text{Tr}_{k+1}(\gamma_N^{(k+1)}), \quad k = 1, \ldots, N-1,$$

for $1 \leq k \leq N-1$, and that $\text{Tr}(\gamma_N^{(k)}) = \|\Phi_N\|_{L^2}^2 = 1$ for all $k < N$.

The $k$-particle marginals satisfy the $N$-particle BBGKY hierarchy

$$i\partial_t \gamma_N^{(k)}(t,x_k;x'_k) = -\sum_{j=1}^{k} (\Delta x_j - \Delta x'_j) \gamma_N^{(k)}(t,x_k;x'_k) + \frac{1}{N} \sum_{1 \leq i < j \leq k} [V_N(x_i - x_j) - V_N(x'_i - x'_j)] \gamma_N^{(k)}(t,x_k;x'_k)$$

$$+ \frac{N - k}{N} \sum_{t=1}^{k} dx_{k+1}[V_N(x_i - x_{k+1}) - V_N(x'_i - x_{k+1})]$$

$$\cdot \gamma_N^{(k+1)}(t,x_k,x_{k+1};x'_k,x_{k+1}).$$

In the limit as $N \rightarrow \infty$, solutions $\gamma_N^{(k)}$ to the BBGKY hierarchy tend to solutions $\gamma^{(k)}$ of the cubic, defocusing GP hierarchy, which we introduce below. In [18, 19, 20, 21], this is obtained from a weak-* limit in the trace class, for $\beta \in (0,1]$. The case $\beta = 1$ covered in [18, 19, 20, 21] is a major achievement that is much harder than $\beta < 1$. Strong convergence in a class of Hilbert-Schmidt type is obtained in [11] for $\beta < \frac{1}{4}$, and subsequently in [12, 13] for $\beta < \frac{2}{5}$.

The cubic GP hierarchy on $\mathbb{R}^3$ for an infinite sequence of bosonic marginal density matrices $(\gamma^{(k)})_{k \in N}$ is given by the infinite system of coupled linear PDE’s

$$i\partial_t \gamma^{(k)} = \sum_{j=1}^{k} [-\Delta x_j, \gamma^{(k)}] + \lambda B_{k+1} \gamma^{(k+1)}, \quad k \in \mathbb{N},$$

for suitable initial data $(\gamma^{(k)}(0))_{k \in N}$, where $\gamma^{(k)}(t;x_k;x'_k)$ is fully symmetric under permutations separately of the components of $x_k := (x_1, \ldots, x_k)$, and of the components of $x'_k := (x'_1, \ldots, x'_k)$. The GP hierarchy is defocusing if $\lambda = 1$, and focusing if
\( \lambda = -1 \) (we are assuming the normalization condition \(|\lambda| = 1\) for simplicity). The interaction term for the \( k \)-particle marginal is defined by

\[
B_{k+1}\gamma^{(k+1)} = B_{k+1}^+\gamma^{(k+1)} - B_{k+1}^-\gamma^{(k+1)}, \quad (1.7)
\]

where

\[
B_{k+1}^+\gamma^{(k+1)} = \sum_{j=1}^{k} B_{j;k+1}^+\gamma^{(k+1)}, \quad (1.8)
\]

and

\[
B_{k+1}^-\gamma^{(k+1)} = \sum_{j=1}^{k} B_{j;k+1}^-\gamma^{(k+1)}, \quad (1.9)
\]

with

\[
\left( B_{j;k+1}^+\gamma^{(k+1)} \right)(t, x_1, \ldots, x_k; x'_1, \ldots, x'_k) = \int dx_{k+1}^{x_k} \delta(x_j - x_{k+1})\delta(x_j' - x_{k+1}')\gamma^{(k+1)}(t, x_1, \ldots, x_{k+1}; x'_1, \ldots, x'_{k+1}) = \gamma^{(k+1)}(t, x_1, \ldots, x_k, x_j; x'_1, \ldots, x'_k, x_j), \quad (1.10)
\]

and

\[
\left( B_{j;k+1}^-\gamma^{(k+1)} \right)(t, x_1, \ldots, x_k; x'_1, \ldots, x'_k) = \int dx_{k+1}^{x_k} \delta(x'_j - x_{k+1})\delta(x'_j - x_{k+1}')\gamma^{(k+1)}(t, x_1, \ldots, x_{k+1}; x'_1, \ldots, x'_{k+1}) = \gamma^{(k+1)}(t, x_1, \ldots, x_k, x'_j; x'_1, \ldots, x'_j, x'_k, x'_j). \quad (1.11)
\]

We say that \( B_{j;k+1}^+ \) contracts the triple of variables \( x_j, x_{k+1}, x'_j \), and that \( B_{j;k+1}^- \) contracts the triple of variables \( x'_j, x_{k+1}, x'_k \).

For \( \alpha \geq 0 \), we define the spaces

\[
\mathcal{S}^\alpha := \left\{ (\gamma^{(k)})_{k \in \mathbb{N}} \big| \text{Tr}(S^{(k,\alpha)}[\gamma^{(k)}]) < M^{2k} \text{ for some constant } M < \infty \right\} \quad (1.12)
\]

where

\[
S^{(k,\alpha)}[\gamma^{(k)}](x_k; x'_k) := \prod_{j=1}^{k} (1 - \Delta x_j)^{\alpha/2}(1 - \Delta x'_j)^{\alpha/2}\gamma^{(k)}(x_k; x'_k). \quad (1.13)
\]

A \textit{mild solution} in the space \( L^\infty_{t \in [0,T]} \mathcal{S}^1 \), to the GP hierarchy with initial data \((\gamma^{(k)}(0))_{k \in \mathbb{N}} \in \mathcal{S}^1\), is a solution of the integral equation

\[
\gamma^{(k)}(t) = U^{(k)}(t)\gamma^{(k)}(0) + i\lambda \int_0^t U^{(k)}(t - s)B_{k+1}\gamma^{(k+1)}(s)ds, \quad k \in \mathbb{N}, \quad (1.14)
\]

satisfying

\[
\sup_{t \in [0,T]} \text{Tr}(S^{(k,1)}[\gamma^{(k)}(t)]) < M^{2k} \quad (1.15)
\]
for a finite constant \( M \) independent of \( k \). Here,

\[ U^{(k)}(t) := \prod_{\ell=1}^{k} e^{it(\Delta x_\ell - \Delta x'_\ell)} \]  

(1.16)

denotes the free \( k \)-particle propagator. We note that

\[ e^{it(\Delta x_\ell - \Delta x'_\ell)} B_{j;k+1}^{\pm} = B_{j;k+1}^{\pm} e^{it(\Delta x_\ell - \Delta x'_\ell)} , \quad \ell \not\in \{j, k+1\} . \]  

(1.17)

That is, any free propagator \( e^{it(\Delta x_\ell - \Delta x'_\ell)} \) commutes with \( B_{j;k+1}^{\pm} \) if the variables \( x_\ell, x'_\ell \) are not affected by \( B_{j;k+1}^{\pm} \).

We remark that given factorized initial data,

\[ \gamma_0^{(k)}(x_k; x'_k) = \prod_{j=1}^{k} \phi_0(x_j) \overline{\phi_0(x'_j)} , \]  

(1.18)

the condition that \( (\gamma^{(k)}(0)) \in \mathcal{S}_1^1 \) is equivalent to

\[ \text{Tr}(|S^{(k,1)}[\gamma^{(k)}(0)]|) = \| \phi_0 \|_{H^1}^{2k} < M^{2k} , \quad k \in \mathbb{N} , \]  

(1.19)

that is, \( \| \phi_0 \|_{H^1} < M \) for some \( M < \infty \). Then, a solution to the GP hierarchy in \( L_{t \in [0,T]}^\infty \mathcal{S}_1^1 \) having these initial data is given by the sequence of factorized density matrices

\[ \gamma^{(k)}(t; x_k; x'_k) = \prod_{j=1}^{k} \phi_t(x_j) \overline{\phi_t(x'_j)} , \]  

(1.20)

if the corresponding 1-particle wave function satisfies the cubic NLS

\[ i \partial_t \phi_t = -\Delta \phi_t + \lambda |\phi_t|^2 \phi_t , \quad \phi_0 \in H^1 . \]  

(1.21)

In this sense, the NLS is interpreted as the mean field description of an infinite system of interacting bosons. The Cauchy problem (1.21) is globally well-posed in the defocusing case \( \lambda = 1 \), and locally well-posed in the focusing case \( \lambda = -1 \). Solutions to (1.21) conserve the \( L^2 \)-mass \( \| \phi_t \|_{L^2_x} = \| \phi_0 \|_{L^2_x} \) and the energy,

\[ E[\phi_t] = \frac{1}{2} \| \nabla_x \phi_t \|_{L^2_x}^2 + \frac{\lambda}{4} \| \phi_t \|_{L^4_x}^4 = E[\phi_0] . \]  

(1.22)

In particular, \( (\gamma^{(k)})_{k \in \mathbb{N}} \in L_{t \in [0,T]}^\infty \mathcal{S}_1^1 \) is equivalent to \( \| \phi_t \|_{L_{t \in [0,T]}^\infty H^1} < M' \) for some finite constant \( M' \).

The uniqueness of solutions to the GP hierarchy in \( L_{t \in [0,T]}^\infty \mathcal{S}_1^1 \) was established by Erdős, Schlein, and Yau in [18, 19, 20, 21]. This is a crucial, and very involved part of their program to derive the cubic defocusing NLS as the mean field description of a bosonic \( N \)-body Schrödinger evolution, as \( N \to \infty \). Their uniqueness proof uses a sophisticated and extensive construction involving Feynman graph expansions, and high dimensional singular integral estimates. A key ingredient in their proof is a powerful combinatorial method that resolves the problem of the factorial growth of number of terms in iterated Duhamel expansions; we outline it in Section 4.1.
Subsequently, Klainerman and Machedon \cite{31} gave a much shorter proof of the uniqueness of solutions to the GP hierarchy satisfying

$$\left\| \left[ \prod_{j=1}^{k} |\nabla x_j| |\nabla x'_j| \right] \gamma^{(k)} \right\|_{\text{HS}} < c^k, \quad k \in \mathbb{N},$$

(1.23)

for the Hilbert-Schmidt norms

$$\|\gamma^{(k)}\|_{\text{HS}} := \left( \text{Tr}(|\gamma^{(k)}|^2) \right)^{1/2} = \left( \int_{\mathbb{R}^{3k} \times \mathbb{R}^{3k}} |\gamma^{(k)}(x_k; x'_k)|^2 dx_k dx'_k \right)^{1/2},$$

(1.24)

but conditional on the a priori assumption that

$$\left\| \left[ \prod_{j=1}^{k} |\nabla x_j| |\nabla x'_j| \right] B_{k+1}^{\pm} \gamma^{(k+1)} \right\|_{L^1_{t \in [0,T)}} < C^k$$

(1.25)

holds for some finite constants $c, C$ independent of $k$. We will refer to (1.25) as the \textit{Klainerman-Machedon condition}. Their approach uses techniques from the analysis of dispersive nonlinear PDEs, together with the combinatorial method of Erdős, Schlein and Yau \cite{15, 18, 19, 20, 21}, which Klainerman and Machedon presented as the “boardgame argument”. Starting with the work \cite{30} for the cubic GP hierarchy on $\mathbb{R}^2$ and $\mathbb{T}^2$, the approach of Klainerman and Machedon was used by various authors for the derivation of the NLS from interacting Bose gases \cite{8, 11, 12, 13, 30, 45, 47}. The method of Klainerman and Machedon also inspired the analysis of the Cauchy problem for the GP hierarchy which was initiated in \cite{9} and continued e.g. in \cite{15, 45}.

The derivation of nonlinear dispersive PDEs, such as the nonlinear Schrödinger (NLS) or nonlinear Hartree (NLH) equations, from many body quantum dynamics is a very active research topic, and has been approached by many authors in a variety of ways; see \cite{15, 18, 19, 20, 21, 22, 30, 41} and the references therein, and also \cite{11, 12, 13, 23, 24, 25, 26, 27, 28, 39, 40}.

This problem is closely related to the phenomenon of Bose-Einstein condensation (BEC) in systems of interacting bosons, which was first experimentally verified in 1995, \cite{6, 14}. For the mathematical study of BEC, we refer to \cite{2, 35, 36, 37, 38} and the references therein.

## 2. Statement of main results

The only currently available proof of unconditional uniqueness of solutions in $L^1_{t \in [0,T)}$ to the cubic GP hierarchy in $\mathbb{R}^3$ is given in the celebrated works of Erdős, Schlein, and Yau, \cite{18, 19, 20, 21}, using an involved construction based on Feynman graph expansions and high-dimensional singular integral estimates. The purpose of the paper at hand is to present a new, simpler proof. We note that this paper contains several extensive example calculations and detailed explanations of background material for the benefit of the reader, but the actual core of our proof, given in in Sections 7 and 8, is short. We expect that our methods can be extended...
to solutions in $L^\infty_{t \in [0,T)} \mathcal{S}$ for some values $s < 1$, and to GP hierarchies in $\mathbb{R}^n$ with $n$ other than 3.

2.1. Prerequisites. A key tool in our proof is the quantum de Finetti theorem, which is a quantum analogue of the Hewitt-Savage theorem in probability theory, [16]. The strong version is due to Hudson-Moody, and Stormer, [17, 44], and applies to sequences of density matrices that are admissible, i.e.,

$$\gamma^{(k)} = \text{Tr}_{k+1}(\gamma^{(k+1)}) \quad \forall k \in \mathbb{N},$$

similarly to (1.4). We quote it in the formulation presented by Lewin, Nam and Rougerie in [34] ([17, 44] state it in the $C^*$-algebraic context).

**Theorem 2.1.** (Strong Quantum de Finetti theorem, [17, 44, 34]) Let $H$ be any separable Hilbert space and let $H^k = \bigotimes_{\text{sym}} H$ denote the corresponding bosonic $k$-particle space. Let $\Gamma$ denote a collection of admissible bosonic density matrices on $H$, i.e.,

$$\Gamma = (\gamma^{(1)}, \gamma^{(2)}, \ldots)$$

with $\gamma^{(k)}$ a non-negative trace class operator on $H^k$, and $\gamma^{(k)} = \text{Tr}_{k+1}(\gamma^{(k+1)})$, where $\text{Tr}_{k+1}$ denotes the partial trace over the $(k+1)$-th factor. Then, there exists a unique Borel probability measure $\mu$, supported on the unit sphere $S \subset H$, and invariant under multiplication of $\phi \in H$ by complex numbers of modulus one, such that

$$\gamma^{(k)} = \int d\mu(\phi) |\phi\rangle \langle \phi|^\otimes k, \quad \forall k \in \mathbb{N}. \quad (2.2)$$

The limiting hierarchies of marginal density matrices obtained, for each value of the time variable $t$, via weak-* limits from the BBGKY hierarchy of bosonic $N$-body Schrödinger systems as in [18, 19, 20, 21] do not necessarily satisfy admissibility. A weak version of the quantum de Finetti theorem then still applies; in the form quoted in Theorem 2.2 below, it was recently proven by Lewin, Nam and Rougerie [34]. Previously, Ammari and Nier proved an equivalent result in [3, 4] in the context of $N$-body boson systems as $N \to \infty$, with less singular interactions than those considered for the GP hierarchy.

**Theorem 2.2.** (Weak Quantum de Finetti theorem, [34, 3, 4]) Let $H$ be any separable Hilbert space and let $H^k = \bigotimes_{\text{sym}} H$ denote the corresponding bosonic $k$-particle space. Assume that $\gamma^{(N)}$ is an arbitrary sequence of mixed states on $H^N$, $N \in \mathbb{N}$, satisfying $\gamma^{(N)} \geq 0$ and $\text{Tr}_{H^N}(\gamma^{(N)}) = 1$, and assume that its $k$-particle marginals have weak-* limits

$$\gamma^{(k)} := \text{Tr}_{k+1,\ldots,N}(\gamma^{(N)}) \rightharpoonup \ast \gamma^{(k)} \quad (N \to \infty), \quad (2.3)$$

in the trace class on $H^k$ for all $k \geq 1$ (here, $\text{Tr}_{k+1,\ldots,N}(\gamma^{(N)})$ denotes the partial trace in the $(k+1)$-st up to $N$-th component). Then, there exists a unique Borel probability measure $\mu$ on the unit ball $B \subset H$, and invariant under multiplication of $\phi \in H$ by complex numbers of modulus one, such that (2.2) holds for all $k \geq 0$.

In our case, we consider the Hilbert space $H = L^2(\mathbb{R}^3)$. For a detailed discussion of the strong and weak quantum de Finetti theorem, we refer to [34], where the notions of strong and weak quantum de Finetti were introduced.
2.2. Main results. Our main result is a new proof of the unconditional uniqueness of solutions to the GP hierarchy in \(L^\infty_{t\in[0,T]}\mathcal{S}\). We note that the property \(\gamma^{(k)}_{k\in\mathbb{N}} \in L^\infty_{t\in[0,T]}\mathcal{S}\) implies that \(\gamma^{(k)}(t) = \int d\mu(t)(|\phi\rangle\langle\phi|)^{\otimes k}\), where the measure \(\mu\) has bounded support in \(H^1(\mathbb{R}^3)\); this is explained in Lemma 4.5 below.

**Theorem 2.3.** Let \(\gamma^{(k)}(t)_{k\in\mathbb{N}}\) be a mild solution in \(L^\infty_{t\in[0,T]}\mathcal{S}\) to the (de)focusing cubic GP hierarchy in \(\mathbb{R}^3\) with initial data \(\gamma^{(k)}(0)_{k\in\mathbb{N}} \in \mathcal{S}\), which is either admissible, or obtained at each \(t\) from a weak-* limit as described in Theorem 2.2.

Then, \(\gamma^{(k)}_{k\in\mathbb{N}}\) is the unique solution for the given initial data.

Moreover, assume that the initial data \(\gamma^{(k)}(0)_{k\in\mathbb{N}} \in \mathcal{S}\) satisfy
\[
\gamma^{(k)}(0) = \int d\mu(t)(|\phi\rangle\langle\phi|)^{\otimes k}, \quad \forall k \in \mathbb{N},
\]
(as guaranteed by Theorems 2.1 and 2.2) where \(\mu\) is a Borel probability measure supported either on the unit sphere or on the unit ball in \(L^2(\mathbb{R}^3)\), and invariant under multiplication of \(\phi \in \mathcal{H}\) by complex numbers of modulus one. Then,
\[
\gamma^{(k)}(t) = \int d\mu(t)(|S_t(\phi)\rangle\langle S_t(\phi)|)^{\otimes k}, \quad \forall k \in \mathbb{N},
\]
where \(S_t : \phi \mapsto \phi_t\) is the flow map of the cubic (de)focusing NLS, for \(t \in [0,T]\). That is, \(\phi_t\) satisfies (1.21) with initial data \(\phi\).

Our proof of uniqueness uses the Erdös-Schlein-Yau combinatorial method [18 19 20 21], in boardgame form as presented by Klainerman-Machedon in [31]. However, we do not use the Klainerman-Machedon condition (1.25), but instead apply the quantum de Finetti theorems. The uniqueness established in Theorem 2.3 is unconditional, and equivalent to the one proven in the celebrated works of Erdös, Schlein and Yau, [18 19 20 21].

2.2.1. Uniqueness of strong solutions. Our next result addresses the uniqueness of strong solutions, and shows the strength of de Finetti for the GP hierarchy. We consider strong solutions to the GP hierarchy \(\Gamma \in C^1([0,T],\mathcal{S})\) with \(\partial_t \Gamma \in C([0,T],\mathcal{S}^{-1})\) in \(\mathbb{R}^d\) where \(d = 1,2,3\) (see [7] for a discussion on the level of the NLS). The quantum de Finetti theorem can be used for a short direct proof of uniqueness in this case, along the lines of Spohn’s argument for the Vlasov hierarchy [43].

**Theorem 2.4.** Let \(d \in \{1,2,3\}\), and \(\Gamma = (\gamma^{(k)})\). Let \(\hat{\Delta} \Gamma := (\sum_{j=1}^{k}[\Delta_{x_j},\gamma^{(k)}])_{k\in\mathbb{N}}\) and \(\hat{B} \Gamma := (B_{k+1}\gamma^{(k+1)})\). Let \(\Gamma(t) = (\gamma^{(k)}(t))\) be a strong solution of the GP hierarchy in \(C^1([0,T],\mathcal{S})\) on \(\mathbb{R}^d\), which is either admissible, or obtained at each \(t\) from a weak-* limit as described in Theorem 2.2. Then,
\[
i\partial_t \Gamma(t) = \hat{L} \Gamma(t) \in C([0,T],\mathcal{S}^{-1}) \quad , \quad \hat{L} = -\hat{\Delta} + \lambda \hat{B} \quad , \quad \lambda \in \{1,-1\},
\]
and \(\Gamma(t)\) is the unique solution for the given initial data.
3. Proof of Theorem 2.4

Let $\Gamma_\phi := ((|\phi\rangle\langle\phi|)^{\otimes k})_{k \in \mathbb{N}}$ for brevity. We know that $\int d\nu(\phi) \Gamma_{S_t(\phi)}$ solves (2.6), with $\Gamma_0 = \int d\nu(\phi) \Gamma_\phi$, if $S_t(\phi)$ is the flow map of the cubic (de)focusing NLS

$$i\partial_t \phi_t = -\Delta \phi_t + \lambda |\phi_t|^2 \phi_t$$

(3.1)

with initial data $\phi_0 \in H^1$. Since by assumption, $\Gamma(t)$ is either admissible, or obtained at each $t$ from a weak-* limit as described in Theorem 2.2, the de Finetti theorem implies that for every $t \geq 0$ there exists a Borel probability measure $\mu_t$ on $L^2(\mathbb{R}^d)$ such that

$$\Gamma(t) = \int d\mu_t(\phi) \Gamma_\phi.$$  

(3.2)

Furthermore, we note that for any $s \geq 0$ and an arbitrary small $\eta > 0$,

$$\text{Tr}(\big| S^{(k,s-2)}[(\hat{\Gamma})^{(k)}]_{\phi} \big|) \leq C \text{Tr}(\big| S^{(k,s)}[\Gamma^{(k)}]_{\phi} \big|),$$

(3.3)

This inequality was proven in [10] (it corresponds to a generalized Sobolev inequality for density matrices). Moreover, it is evident that for any $s \geq 0$,

$$\text{Tr}(\big| S^{(k,s-2)}[\hat{\Gamma}^{(k)}]_{\phi} \big|) \leq \text{Tr}(\big| S^{(k,s)}[\Gamma^{(k)}]_{\phi} \big|), \quad k \in \mathbb{N}.$$  

(3.4)

Since $d^2 + \eta \leq 2$ for $d \leq 3$ in (3.3), we find that by setting $s = 1$ in (3.3) and (3.4),

$$\text{Tr}(\big| S^{(k,0)}[\hat{\Gamma}^{(k)}]_{\phi} \big|) < C \text{Tr}(\big| S^{(k)}[\Gamma^{(k)}]_{\phi} \big|) < C^k, \quad k \in \mathbb{N},$$

(3.5)

or in other words, $\hat{\Gamma} \in C([0,T), \hat{\mathcal{S}}^{-1})$. Here, the second inequality follows from $\Gamma \in C^1([0,T), \mathcal{S}^1)$, see [1.19]. Since $\Gamma(t)$ is a strong solution of (2.6) in $C^1([0,T), \mathcal{S}^1)$, we obtain that

$$i\partial_t \Gamma^{(k)}(t) = \lim_{h \to 0} \frac{1}{h} \left( \Gamma^{(k)}(t+h) - \Gamma^{(k)}(t) \right) = (\hat{\Gamma}^{(k)})(t),$$

(3.6)

hence $\partial_t \Gamma \in C([0,T), \mathcal{S}^{-1})$. Then, indeed,

$$\frac{d}{dt} \int d\mu_t(\phi) \Gamma_\phi = \lim_{h \to 0} \frac{1}{h} \left( \int d\mu_{t+h}(\phi) \Gamma_\phi - \int d\mu_t(\phi) \Gamma_\phi \right) = -i \int \hat{\Gamma}_\phi d\mu_t(\phi)$$

(3.7)

holds in $C([0,T), \mathcal{S}^{-1})$. In this sense, $\mu_t$ is differentiable, with derivative given by the operator $\hat{L}$.

In analogy to Spohn’s argument in [43], we can show that the measure $\mu_t$ induces a flow on the unit ball which satisfies the GP-equation (3.1). To this end, we define

$$\tilde{\Gamma}(t) := \int d\mu_t(\phi) \Gamma_{S_{-t}(\phi)}.$$  

(3.8)
Differentiating this with respect to $t$, gives

\[
\frac{d}{dt} \tilde{\Gamma}(t) = \lim_{h \to 0} \frac{1}{h} \left( \int d\mu_{t+h}(\phi) \Gamma_{S_{t+h}}(\phi) - \int d\mu_t(\phi) \Gamma_{S_t}(\phi) \right)
\]

\[+ \lim_{h \to 0} \frac{1}{h} \int d\mu_{t+h}(\phi) \left( \Gamma_{S_{t+h}}(\phi) - \Gamma_{S_t}(\phi) \right) \]

\[= -i \int d\mu_t(\phi) \tilde{\Gamma}_{S_{t}}(\phi) + i \int d\mu_t(\phi) \tilde{\Gamma}_{S_{t}}(\phi) = 0,
\]

(3.11)

where we applied (3.7) to (3.9) to get the first term in (3.11), and where we applied (3.6) to (3.10) to get the second term in (3.11).

Since the map $\phi \mapsto S_t(\phi)$ is a bijection on the intersection of the unit ball of $L^2(\mathbb{R}^d)$ with $H^1(\mathbb{R}^d)$, we obtain by a simple variable transformation

\[\int d\mu_t(\phi) \Gamma_{S_{t}}(\phi) = \int d\mu_t(S_t(\phi)) \Gamma_{\phi}.
\]

(3.12)

Because of (3.11), we find that

\[\tilde{\Gamma}(t) = \int d\mu_t(S_t(\phi)) \Gamma_{\phi}
\]

(3.13)

does not depend on $t$, and by the uniqueness parts of Theorems 2.1 and 2.2 we infer that

\[d\mu_t(S_t(\phi)) = d\mu_0(\phi).
\]

Hence, by variable transformation,

\[\Gamma(t) = \int d\mu_t(\phi) \Gamma_{\phi} = \int d\mu_t(S_t(\phi)) \Gamma_{S_t}(\phi) = \int d\mu_0(\phi) \Gamma_{S_t}(\phi).
\]

By the uniqueness parts of Theorems 2.1 and 2.2, $d\nu = d\mu_0$.

\[\square
\]

4. Proof of Theorem 2.3

The remainder of this paper is dedicated to the proof of Theorem 2.3. As a preparation, we present three auxiliary combinatorial tools used for the organization of Duhamel expansion terms, in Sections 4.1, 5, and 6. For the convenience of the reader, we will give a detailed survey of background material in these parts, and present several detailed example calculations. The core of our proof is contained in Sections 7 and 8 and is short.

4.1. The Erdős-Schlein-Yau combinatorial method in boardgame form.

To prove uniqueness, we will show that the trace norm of $\gamma^{(k)}$ (instead of the Hilbert-Schmidt norm of $S^{(k,1)}[\gamma^{(k)}]$ as in [31]) is zero if the initial data is zero (see Section 4.2 for details). We will use the powerful combinatorial method of Erdős, Schlein and Yau [18, 19, 20, 21], which was presented in an elegant and accessible form by Klainerman and Machedon in [31] as a “boardgame argument”.

\[\square
\]
To begin with, we consider the $r$-fold iterate of the Duhamel formula (1.1) for $\gamma^{(k)}$, with initial data $\gamma_0^{(k)} = 0$, for some arbitrary $r \in \mathbb{N}$,
\[
\gamma^{(k)}(t) = (i\lambda)^r \int_{t\geq t_1\geq \cdots \geq t_r} dt_1 \cdots dt_r U^{(k)}(t-t_1) B_{k+1} U^{(k+1)}(t_1-t_2) \cdots \\
\cdots U^{(k+r-1)}(t_{r-1}-t_r) B_{k+r} \gamma^{(k+r)}(t_r)
\]
\[
= \int_{t\geq t_1\geq \cdots \geq t_r} dt_1 \cdots dt_r J^k(t_r), \quad t_r := (t_1, \ldots, t_r). \quad (4.1)
\]
We will prove the following main lemma.

**Lemma 4.1.** Assume that $(\gamma^{(k)}(t))$ is a mild solution to the cubic GP hierarchy (1.6) with initial data $\gamma^{(k)}(0) = 0$ for all $k$, which is either admissible or obtained at each $t$ from a weak-* limit as in Theorem 2.2. Moreover, assume that
\[
\sup_{t \in [0,T)} \text{Tr}(\|S^{(k,1)}[\gamma^{(k)}(t)]\|) < M^{2k}, \quad k \in \mathbb{N}, \quad (4.2)
\]
holds for some finite constant $M$ independent of $k$ and $t$. Then, for $t \in [0,T)$, the estimate
\[
\text{Tr}(\|\gamma^{(k)}(t)\|) < 2 M^{2k} (2CM^4T)^{(r+1)/2} \quad (4.3)
\]
holds. In particular, the right hand side converges to zero in the limit as $r \to \infty$ for $T < (2CM^4)^{-1}$ (independent of $k$), for every $k \in \mathbb{N}$.

This main lemma implies that
\[
\text{Tr}(\|\gamma^{(k)}(t)\|) = 0, \quad t \in [0,T), \quad (4.4)
\]
and thus that $\gamma^{(k)}(t) = 0$ for $t \in [0,T)$. Hence, uniqueness holds. That the trace norm of the right hand side converges to zero as $r \to \infty$, for $t \in [0,T)$ and $T > 0$ sufficiently small (where the smallness condition on $T$ is uniform in $k$).

A key difficulty in this approach stems from the fact that the interaction operator $B_{k+1}$ is the sum of $O(\ell)$ terms, therefore (4.1) contains $O\left(\frac{(k+r-1)!}{(k-1)!}\right) = O(r!)$ terms. The boardgame argument allows one to control this rapid increase of the number of terms as $r \to \infty$, using the fact that the ordered time integrals $t_1 \geq t_2 \geq \cdots \geq t_r$ extends over a simplex of volume $O(\frac{1}{r!})$. We give a short summary of the method.

Recalling that $B_{k+1} = \sum_{j=1}^\ell B_{j;\ell+1}$, we write
\[
J^k(t_r) = \sum_{\rho \in \mathcal{M}_{k,r}} J^k(\rho; t_r), \quad (4.5)
\]
where
\[
J^k(\rho; t_r) := (i\lambda)^r U^{(k)}(t-t_1) B_{\rho(k+1),k+1} U^{(k+1)}(t_1-t_2) \cdots \\
\cdots U^{(k+\ell-1)}(t_{\ell-1}-t_{\ell}) B_{\rho(k+\ell),k+\ell} \cdots U^{(k+r-1)}(t_{r-1}-t_r) B_{\rho(k+r),k+r} \gamma^{(k+r)}(t_r), \quad (4.6)
\]
and $\rho$ is a map
\[
\rho: \{k+1,r+2,...,k+r\} \rightarrow \{1,2,...,k+r-1\}, \\
\rho(2) = 1, \quad \rho(j) < j \quad \forall j. \quad (4.7)
\]
Here $\mathcal{M}_{k,r}$ denotes the set of all such mappings $\rho$. 


We observe that each map \( \rho \) can be represented by highlighting one nonzero entry \( B_{\rho(k+\ell),k+\ell} \) in each column of an \((k+r-1) \times r\) matrix \([A_{i,\ell}]\) with entries \( A_{i,\ell} = B_{i,k+\ell} \) for \( i < k + \ell \), and \( A_{i,\ell} = 0 \) for \( i \geq k + \ell \). As an example, consider

\[
\begin{pmatrix}
B_{1,k+1} & B_{1,k+2} & \cdots & \cdots & \cdots & B_{1,k+r} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
B_{k,k+1} & B_{k,k+2} & \cdots & \cdots & \cdots & \vdots \\
0 & B_{k+1,k+2} & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & B_{k+r-1,k+r}
\end{pmatrix}.
\]

(4.8)

Then,

\[
\gamma^{(k)}(t) = \sum_{\rho \in M_{k,r}} \int_{t \geq t_1 \geq \cdots \geq t_r} J^k(\rho; t) \ dt_1 \cdots dt_r,
\]

where the time domains are given by the same simplex \( \{ t > t_1 > \cdots > t_r \} \subset [0, t]^r \) for all integrals in the sum over \( \rho \).

We next consider the integrals with permuted time integration orders

\[
I(\rho, \pi) = \int_{\{ t_{\pi^{-1}(1)} \geq \cdots \geq t_{\pi^{-1}(r)} \}} J^k(\rho; t) \ dt_1 \cdots dt_r,
\]

(4.10)

where \( \pi \) is a permutation of \( \{1, 2, \ldots, r\} \). This corresponds to replacing the simplex \( \{ t > t_1 > \cdots > t_r \} \subset [0, t]^r \) by an isometric image in \([0, t]^r\). One can associate to \( I(\rho, \pi) \) the matrix

\[
\begin{pmatrix}
t_{\pi^{-1}(1)} & t_{\pi^{-1}(2)} & \cdots & t_{\pi^{-1}(r)} \\
B_{1,k+1} & B_{1,k+2} & \cdots & B_{1,k+r} \\
\vdots & \vdots & \ddots & \vdots \\
B_{k,k+1} & B_{k,k+2} & \cdots & \vdots \\
0 & B_{k+1,k+2} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_{k+r-1,k+r}
\end{pmatrix}
\]

whose columns are labeled 1 through \( r \) and whose rows are labeled 0, 1, \ldots, \( k+r-1 \), and where the highlighted entries correspond to \( B_{\rho(k+\ell),k+\ell} \).

Using the combinatorial method in \cite{18, 19, 20, 21} in the form presented in \cite{31}, a board game is introduced on the set of such matrices. A acceptable move is characterized as follows: If \( \rho(k+\ell) < \rho(k+\ell+1) \), the player is allowed to do the following three changes at the same time:

- exchange the highlights in columns \( \ell \) and \( \ell + 1 \),
- exchange the highlights in rows \( k + \ell - 1 \) and \( k + \ell \),
- exchange \( t_{\pi^{-1}(\ell)} \) and \( t_{\pi^{-1}(\ell+1)} \).
We note that the rows \( k + \ell \) and \( k + \ell + 1 \) do not necessarily contain highlights. A main property of the integrals \( I(\rho, \pi) \) is invariance under acceptable moves, \[18, 19, 20, 21, 31\]:

**Lemma 4.2.** If \((\rho, \pi)\) is transformed into \((\rho', \pi')\) by an acceptable move, then \( I(\rho, \pi) = I(\rho', \pi') \).

We say that a matrix of the type (4.8) is in upper echelon form if each highlighted entry in a row is to the left of each highlighted entry in a lower row. For example, the following matrix is in upper echelon form (with \( k = 1 \) and \( r = 4 \)):

\[
\begin{bmatrix}
B_{1,2} & B_{1,3} & B_{1,4} & B_{1,5} \\
0 & B_{2,3} & B_{2,4} & B_{2,5} \\
0 & 0 & B_{3,4} & B_{3,5} \\
0 & 0 & 0 & B_{4,5}
\end{bmatrix}.
\]

Then, the following normal form property holds, \[18, 19, 20, 21, 31\]:

**Lemma 4.3.** For each matrix in \( M_{k,r} \), there is a finite number of acceptable moves that transforms the matrix into upper echelon form. Moreover, let \( C_{k,r} \) denote the number of upper echelon matrices of size \((k + r - 1) \times r\). Then,

\[
C_{k,r} \leq 2^{k+r}.
\]

Let \( N_{k,r} \) denote the subset of matrices in \( M_{k,r} \) which are in upper echelon form. Let \( \sigma \) account for a matrix in \( N_{k,r} \). We write \( \rho \sim \sigma \) if the matrix corresponding to \( \rho \) can be transformed into that corresponding to \( \sigma \) in finitely many acceptable moves. We note that \( \sigma \) satisfies the same properties (4.7) as \( \rho \), but in addition,

\[
\sigma(j) \leq \sigma(j'), \quad \forall j < j'.
\]

Then, the following key theorem holds, \[18, 19, 20, 21, 31\]:

**Theorem 4.4.** Suppose \( \sigma \in N_{k,r} \). Then, there exists a subset of \([0,t]^r\), denoted by \( D(\sigma, t) \), such that

\[
\sum_{\rho \sim \sigma} \int_{t'_1 \geq \cdots \geq t'_r} J^k(\rho; t'_r) \ dt'_1 \cdots dt'_r = \int_{D(\sigma, t)} J^k(\sigma; t'_r) \ dt'_1 \cdots dt'_r.
\]

4.2. **Setup of the proof.** We now give a precise formulation of the framework in which we will prove Theorem 2.3. Let us assume that we have two positive semidefinite solutions \((\gamma_j^{(k)}(t))_{k \in \mathbb{N}} \in L^\infty_{t \in [0,T]} \mathcal{S}_1^j\) satisfying the same initial data, \((\gamma_1^{(k)}(0))_{k \in \mathbb{N}} = (\gamma_2^{(k)}(0))_{k \in \mathbb{N}} \in \mathcal{S}_1^1\). Then,

\[
\gamma^{(k)}(t) := \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t), \quad k \in \mathbb{N},
\]
is a solution to the GP hierarchy with initial data $\gamma^{(k)}(0) = 0 \forall k \in \mathbb{N}$, and it suffices to prove that $\gamma^{(k)}(t) = 0 \forall k \in \mathbb{N}$, and for all $t \in [0, T]$. This is due to the linearity of the GP hierarchy.

We note that $\gamma^{(k)}$, as a difference of positive semidefinite marginal density matrices, is not in general positive semidefinite.

From the assumptions of Theorem 2.3, we have that
\[
\sup_{t \in [0, T]} \text{Tr}([S^{(k,1)}]^{(k)}(t)]) < M^{2k}, \quad k \in \mathbb{N}, \quad i = 1, 2, \tag{4.15}
\]
for some finite constant $M$ independent of $k$ and $t$.

To ensure the applicability of Theorems 2.1 and 2.2, we note that if
\[
\gamma^{(k)}_j = \text{Tr}_{k+1}(\gamma^{(k+1)}_j) \quad \forall k \in \mathbb{N}, \quad j = 1, 2, \tag{4.16}
\]
are admissible, it follows immediately that $(\gamma^{(k)})_{k \in \mathbb{N}}$ is admissible. Moreover if both $(\gamma^{(1)})$ and $(\gamma^{(2)})$ are obtained from a weak-* limit, then so is $(\gamma^{(k)})$.

Thus, from Theorems 2.1 and 2.2, we have that
\[
\gamma^{(k)}(t) = \int d\mu_t^{(j)}(\phi)(\langle \phi \rangle \langle \phi \rangle)^{\otimes k}, \quad j = 1, 2, 
\]
\[
\gamma^{(k)}(t) = \int d\mu_t(\phi)(\langle \phi \rangle \langle \phi \rangle)^{\otimes k}, \tag{4.17}
\]
where $\tilde{\mu}_t := \mu_t^{(1)} - \mu_t^{(2)}$ is the difference of two probability measures on the unit ball in $L^2(\mathbb{R}^3)$. We remark that (4.15) is equivalent to
\[
\int d\mu_t^{(j)}(\phi)||\phi||_{H^1}^{2k} < M^{2k}, \quad j = 1, 2, \tag{4.18}
\]
for all $k \in \mathbb{N}$, where $H^1 = \{f \in L^2(\mathbb{R}^3) \mid ||\langle \nabla \rangle f||_{L^2} < \infty\}$, and $\langle \nabla \rangle := \sqrt{1 - \Delta}$.

**Lemma 4.5.** Let $\mu$ be a Borel probability measure in $L^2(\mathbb{R}^3)$, and assume that
\[
\int d\mu(\phi)||\phi||_{H^1}^{2k} \leq M^{2k} \tag{4.19}
\]
holds for some finite constant $M > 0$, and all $k \in \mathbb{N}$. Then,
\[
\mu\left(\left\{ \phi \in L^2(\mathbb{R}^3) \mid ||\phi||_{H^1} > M \right\}\right) = 0. \tag{4.20}
\]

**Proof.** From Chebyshev’s inequality, we have that
\[
\mu\left(\left\{ \phi \in L^2(\mathbb{R}^3) \mid ||\phi||_{H^1} > \lambda \right\}\right) \leq \frac{1}{\lambda^{2k}} \int d\mu(\phi)||\phi||_{H^1}^{2k} \leq \frac{M^{2k}}{\lambda^{2k}} \tag{4.21}
\]
for any $k \geq 0$. Evidently, for $\lambda > M$, the r.h.s. tends to zero when $k \to \infty$. \hfill $\square$

From here on, we will consider the representation of the expansion (4.1) for $\gamma^{(k)}(t)$ in upper echelon normal form, given by the right hand side of (4.13). Then,
\[
\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{N}_{k,r}} \int_{D(\sigma)} dt_1 \cdots dt_r U^{(k)}(t - t_1) B_{\sigma(k+1),k+1} U^{(k+1)}(t_1 - t_2) \cdots \\
\cdots U^{(k+r-1)}(t_{r-1} - t_r) B_{\sigma(k+r),k+r} \gamma^{(k+r)}(t_r). \tag{4.22}
\]
The sum with respect to $\sigma$ extends over all inequivalent upper echelon forms.

Using the quantum de Finetti theorem (by which we henceforth refer to either the strong or the weak version), we obtain

$$
\gamma^{(k)}(t) = \sum_{\sigma \in N_{k,r}} \int_{D(\sigma,t)} dt_1, \ldots, dt_r \int d\tilde{\mu}_r(\phi) J^k(\sigma; t, t_1, \ldots, t_r),
$$

(4.23)

where

$$
J^k(\sigma; t, t_1, \ldots, t_r; x_k; x'_k) = \left(U^{(k)}(t-t_1)B_{\sigma(k+1),k+1}U^{(k+1)}(t_1-t_2)\cdotsight.
$$

$$
\cdots B_{\sigma(k+\ell),k+\ell}U^{(k+\ell)}(t_\ell-t_{\ell+1})B_{\sigma(k+\ell+1),k+\ell+1}U^{(k+\ell+1)}(t_{\ell+1}-t_{\ell+2})\cdots
$$

$$
\cdots U^{(k+r-1)}(t_{\ell-1}-t_{\ell})B_{\sigma(k+r),k+r}((\phi)\langle \phi \rangle^{(k+r)})^\otimes(\ell;\ell)\left(x_k; x'_k\right).
$$

(4.24)

Here, we may think of the time variable $t_\ell$ as being attached to the interaction operator $B_{\sigma(k+\ell),k+\ell}$. For fixed $\phi$, we note that since

$$
(\langle \phi \rangle \langle \phi \rangle)^{\otimes(k+r)}(x_k; x'_k) = \prod_{i=1}^{k+r} (\langle \phi \rangle(\sigma_i))(x_i; x'_i)
$$

(4.25)

is given by a product of 1-particle kernels, it follows that

$$
J^k(\sigma; t, t_1, \ldots, t_r; x_k; x'_k) = \prod_{j=1}^{k} J_{j}^{(\sigma_j; t, t_{\ell_j,1}, \ldots, t_{\ell_j,m_j}; x_j; x'_j)}
$$

(4.26)

likewise holds product form, for each fixed $\sigma$. This is because in (4.24), the operators $B_{\sigma(k+\ell),k+\ell}$ and $U^{(k+\ell)}(t_\ell-t_{\ell+1})$ each map products of 1-particle kernels to products of 1-particle kernels (but the operators $B_{\sigma(k+\ell),k+\ell}$ do in general not preserve positive semidefiniteness). Each 1-particle kernel $J_{j}^{(\sigma)}$ can be written as a Duhamel expansion in itself, with interaction operators inherited from those appearing in $J^k$.

We label the interaction operators in $J_{j}^{(\sigma)}$ "internally" with $\sigma_j$, $j = 1, \ldots, k$, (which are automatically in upper echelon form relative to $J_{j}^{(\sigma)}$). More details are given in Section 4 below.

For a fixed $k$, the number of inequivalent echelon forms is bounded by $C^r$, using Lemma 4.3. Hence,

$$
\text{Tr}(|\gamma^{(k)}|) \leq C^r \sum_{i=1,2} \sup_{\sigma} \int_{[0,t]} dt_1 \cdots dt_r \int d\tilde{\mu}_r(\phi) \prod_{j=1}^{k} \text{Tr}\left(|J_{j}^{(\sigma_j; t, t_{\ell_j,1}, \ldots, t_{\ell_j,m_j})}|\right).
$$

(4.27)

The time variable $t_{\ell_j,\alpha}$ corresponds to the one attached to the $\alpha$-th interaction operator (counting from the left) appearing in the factor $J_{j}^{(\sigma)}$ (every $t_{\ell_j,\alpha}$ corresponds uniquely to one of the time variables $t_\ell$ in (4.24)). Based on the expression (4.27), our goal is to prove the estimate

$$
\text{Tr}(|\gamma^{(k)}(t)|) < 2 M^{2k-2} (2CM^4T)^{(r+1)/2}
$$

(4.28)

asserted in Lemma 4.1. It implies that for any $k \in \mathbb{N}$, the right hand side tends to zero as $r \to \infty$, for $t \in [0, T)$, and sufficiently small $T > 0$ (independent of $k$). Since $r$ is arbitrary, this implies that the left hand side equals zero, thus establishing uniqueness. By iterating this argument on the union of intervals $[0, T) \cup [T, 2T) \cup \ldots$, we conclude.
uniqueness extends to the entire time of existence for a given solution. We note that in (4.27), the distinction between focusing and defocusing GP hierarchy has disappeared, since $|\lambda| = 1$ in both cases.

5. Binary tree graphs

Because every interaction operator contracts precisely two factors, one can conveniently organize the above expansions for $J^k$ and $J^1$ with the help of binary tree graphs, for arbitrary values of $k$ and $r$. For the convenience of the reader, we first discuss the factorization (4.26) with an example.

5.1. An example for $k = 3$. As an example, let $k = 3$ and $r = 4$, and let us consider

$$J^3(\sigma; t, t_1, \ldots, t_4) = U_{0,1}^{(3)} B_{2,4} U_{1,2}^{(4)} B_{2,3} U_{2,3}^{(5)} B_{3,6} U_{3,4}^{(6)} B_{5,7} (|\phi\rangle \langle \phi|)^{\otimes 7}$$  
(5.1)

where $U_{i,i'}^{(j)} := U_{i,i'}^{(j)}(t_i - t_{i'})$ with $t_0 := t$, and where $\sigma$ corresponds to the upper echelon matrix

$$
\begin{bmatrix}
 B_{1,4} & B_{1,5} & B_{1,6} & B_{1,7} \\
 B_{2,4} & B_{2,5} & B_{2,6} & B_{2,7} \\
 B_{3,4} & B_{3,5} & B_{3,6} & B_{3,7} \\
 0 & B_{4,5} & B_{4,6} & B_{4,7} \\
 0 & 0 & B_{5,6} & B_{5,7} \\
 0 & 0 & 0 & B_{6,7}
\end{bmatrix}.
$$  
(5.2)

We now read off from this matrix which terms are “connected” via contractions, starting with the rightmost interaction operator. Although all 7 factors in $(|\phi\rangle \langle \phi|)^{\otimes 7}$ are indistinguishable, we enumerate the factors, and write the product in the form $\otimes_{i=1}^7 u_i$, ordered with increasing index $i$ (where $u_i = |\phi\rangle \langle \phi|$ for every $i = 1, \ldots, 7$).

- Clearly, $B_{5,7}$ contracts the factors $u_5$ and $u_7$, and acts trivially as the identity on all other factors $u_i$,

$$B_{5,7}(\otimes_{i=1}^7 u_i) = (\otimes_{i=1}^4 u_i) \otimes \Theta_4 \otimes u_6,$$

where

$$\Theta_4 := B_{1,2}(u_5 \otimes u_7).$$  
(5.4)

The index $\alpha$ in $\Theta_{\alpha}$ associates it to the $\alpha$-th interaction operator from the left in (5.1) (in case of $\Theta_4$, the 4th interaction operator is given by $B_{5,7}$).

- The interaction operator $B_{3,6}$ contracts $U_{3,4}^{(3)} u_3$ and $U_{3,4}^{(1)} u_6$, while it leaves all remaining factors untouched. In particular, it does not affect $\Theta_4$.

$$B_{3,6} U_{3,4}^{(6)} (5.3) = (U_{3,4}^{(2)}(u_1 \otimes u_2)) \otimes \Theta_3 \otimes (U_{3,4}^{(1)} u_4) \otimes (U_{3,4}^{(1)} \Theta_4)$$

where

$$\Theta_3 := B_{1,2}(U_{3,4}^{(2)}(u_3 \otimes u_6)).$$  
(5.6)
The interaction operator $B_{2,5}$ contracts $U_{2,4}^{(1)} u_2$ with $U_{2,4}^{(1)} \Theta_4$ (where we used the group property $U_{2,3}^{(j)} U_{3,4}^{(j)} = U_{2,4}^{(j)}$) corresponding to the 2nd and 5th factor in (5.5), while it leaves all remaining factors untouched.

\[ B_{2,5} U_{2,3}^{(5)} = (U_{2,4}^{(1)} u_1) \otimes \Theta_2 \otimes (U_{2,3}^{(1)} \Theta_3) \otimes (U_{2,4}^{(1)} u_4) \quad (5.7) \]

where

\[ \Theta_2 := B_{1,2}((U_{2,4}^{(1)} u_2) \otimes (U_{2,4}^{(1)} \Theta_4)). \quad (5.8) \]

Finally, the interaction operator $B_{2,4}$ contracts $U_{1,2}^{(1)} \Theta_2$ with $(U_{1,4}^{(1)} u_4)$ corresponding to the 2nd and 4th factor in (5.7), while leaving all other factors untouched,

\[ B_{2,4} U_{1,2}^{(4)} = (U_{1,4}^{(1)} u_1) \otimes \Theta_1 \otimes (U_{1,4}^{(1)} \Theta_3) \quad (5.9) \]

where

\[ \Theta_1 := B_{1,2}((U_{1,2}^{(1)} \Theta_2) \otimes (U_{1,4}^{(1)} u_4)). \quad (5.10) \]

In conclusion, we have found the factorized expression for (5.1).

\[ J^3 = \frac{(U_{0,4}^{(1)} u_1) \otimes (U_{0,1}^{(1)} \Theta_1) \otimes (U_{0,3}^{(1)} \Theta_3)}{J_1^1 = J_2^2 = J_3^3}. \quad (5.11) \]

We may now write the factors $J_j$ as one-particle matrices, and substitute back $u_i = |\phi\rangle \langle \phi|$ for all $i = 1, \ldots, 7$.

- $J_1^1$ corresponds to a free propagation without any interaction operators,

\[ J_1^1 = U_{0,4}^{(1)} |\phi\rangle \langle \phi| \quad (5.12) \]

- Moreover,

\[ J_2^2 = U_{0,1}^{(1)} B_{1,2} U_{1,2}^{(2)} B_{1,3} U_{2,4}^{(1)} B_{3,4} (|\phi\rangle \langle \phi|)^{\otimes 4} \quad (5.13) \]

where the interaction operators correspond to $B_{2,4}$, $B_{2,5}$, $B_{5,7}$; they are re-indexed in a manner that leaves the connectivity structure between contractions invariant. The labeling of interaction operators $B_{\sigma(\ell)}$ here is obtained from a labeling function $\sigma_2$ (corresponding to $\sigma_j$ in (4.26)) where $\sigma_2(2) = 1$, $\sigma_2(3) = 1$, $\sigma_2(4) = 3$.

- Finally,

\[ J_3^3 = U_{0,3}^{(1)} B_{1,2} U_{1,4}^{(2)} (|\phi\rangle \langle \phi|)^{\otimes 2} \quad (5.14) \]

where the interaction operator corresponds to $B_{3,6}$, and can be labeled with $\sigma_3(2) = 1$.

We observe that for $\ell < \ell'$, the interaction operators $B_{\sigma(\ell)}$ and $B_{\sigma(\ell')}$ in $J^3$ (which are highlighted in (5.2)) belong to the same factor $J_j$ if either $\ell = \ell'$, or $\ell = \sigma(\ell')$. In this case, we can think of them as being “connected”; below, we will introduce binary tree graphs that encode this connectivity structure.

This example also illustrates how the internal labeling functions $\sigma_j$ in (4.26) are deduced from the “global” labeling function $\sigma$. In the sense outlined above, we may think of $\sigma_j$ as the restriction of $\sigma$ to $J_j^1$. 

We note that in $J_1^1$ and $J_3^1$, there is a free propagator applied to each $\phi$; this will allow for a straightforward application of Strichartz estimates to control contractions due to interaction operators. However, the term $J_2^1$ involves the contraction of factors $|\phi\rangle\langle\phi|$ without any free propagation term inbetween. We will call the factor $J_1^1$ distinguished while $J_1^1$ and $J_3^1$ are regular.

5.2. Definition of binary trees. We now introduce binary tree graphs as a bookkeeping device to keep track of the complicated contraction structures imposed by the interaction operators inside the iterated Duhamel formula (4.21).

To this end, we associate (4.24) to the union of $k$ disjoint binary tree graphs, $(\tau_j)_{j=1}^k$. We note that these appear as “skeleton graphs” for the more complicated graphs in [18] [19] [20] [21]. We assign:

- An internal vertex $v_\ell$, $\ell = 1, \ldots, r$, to each operator $B_{x(k+\ell),k+\ell}$. Accordingly, the time variable $t_\ell$ in (4.24) is thought of as being attached to the vertex $v_\ell$.
- A root vertex $w_j$, $j = 1, \ldots, k$, to each factor $J_1^1(x_1;\ldots;x_j)$ in (4.25).
- A leaf vertex $u_i, i = 1, \ldots, k+r$, to the factor $|\phi\rangle\langle\phi|(x_i;x'_i)$ in (4.25).

For the sake of concreteness, we draw graphs as follows: We consider the strip in $(x,y) \in \mathbb{R}^2$ given by $x \in [0,1]$. We draw all root vertices $(w_j)_{j=1}^k$, ordered vertically, on the line $x = 0$, all internal vertices $(v_\ell)_{\ell=1}^r$ in the region $x \in (0,1)$, where $v_\ell$ is on the right of $v_\ell'$ if $\ell' > \ell$. Finally, we draw all leaf vertices $(u_i)_{i=1}^{k+r}$, ordered vertically, on the line $x = 1$.

Next, we introduce the equivalence relation “$\sim$” of connectivity between vertices. Between any pair of connected vertices, we draw a connecting line, which we refer to as an edge:

- Let $v_\ell$ be the internal vertex with smallest value of $\ell$ such that $\sigma(\ell) = j$; then, we say that $v_\ell$ is connected to the root vertex $w_j$, that is, $w_j \sim v_\ell$.
- If there is no internal vertex connected to $w_j$, we draw an edge connecting $w_j$ to the leaf vertex $u_j$, and say that they are connected, $w_j \sim u_j$.
- Given $k < \ell \leq k+r$, if there exists $\ell' > \ell$ such that $\ell = \sigma(\ell')$ or $\sigma(\ell) = \sigma(\ell')$, we say that $v_\ell \sim v_{\ell'}$ are connected.

$v_\ell$ is then called a parent vertex of $v_{\ell'}$, and $v_{\ell'}$ is called a child vertex of $v_\ell$. We denote the two child vertices of $v_\ell$ by $v_{\kappa_-(\ell)}$ and $v_{\kappa_+(\ell)}$, using the condition $\kappa_-(\ell) < \kappa_+(\ell)$.

If there exists no internal vertex $v_{\ell'}$ with $\ell' > \ell$ such that $\ell = \sigma(\ell')$, we say that $v_\ell$ is connected to the leaf vertex $u_\ell$, $v_\ell \sim u_\ell$; if there exists no internal vertex $v_{\ell'}$ with $\ell' > \ell$ such that $\sigma(\ell) = \sigma(\ell')$, we say that $v_\ell$ is connected to the leaf vertex $u_{\sigma(\ell)}$, $v_\ell \sim u_{\sigma(\ell)}$. In these cases, $v_\ell$ is the parent vertex of $u_\ell$ (or $u_{\sigma(\ell)}$), and $u_\ell$ (or $u_{\sigma(\ell)}$) is a child vertex of $v_\ell$.

This implies that every internal vertex has precisely two child vertices, which can either be internal or leaf vertices (they do not need to be of the same type). Every root vertex has precisely one child vertex, which could be of internal or of leaf type. Every internal or leaf vertex has exactly one parent vertex.
We conclude that the graph thus obtained is the disjoint union of \( k \) binary trees, which we denote by \( (\tau_j)_{j=1}^k \), where the root of \( \tau_j \) is the root vertex \( w_j \) (if \( w_j \sim u_j \) without internal vertices inbetween, then the binary tree consists trivially only of a single edge connecting one root and one leaf vertex).

We say that the tree \( \tau_j \) is distinguished if \( v_r \in \tau_j \), and regular if \( v_r \notin \tau_j \). We call the two leaf vertices connected to \( v_r \) distinguished leaf vertices, and all others regular leaf vertices. Clearly, there are \( k - 1 \) regular trees, and one distinguished tree in this construction.

![Figure 1. The disjoint union of three tree graphs \( \tau_j \), \( j = 1,2,3 \), corresponding to the example discussed in Section 5.1, for \( k = 3 \), \( r = 4 \), and (5.15).](image)

\[
J^3(\sigma; t, t_1, \ldots, t_4) = U_{0,1}^{(3)}B_{2,4}U_{1,2}^{(4)}B_{2,5}U_{2,3}^{(5)}B_{3,6}U_{3,4}^{(6)}B_{5,7}(|\phi\rangle\langle\phi|)^{\otimes 7}. \quad (5.15)
\]

The root vertex \( w_1 \) belongs to the tree \( \tau_j \), \( j = 1,2,3 \). The internal vertices correspond to \( v_1 \sim B_{2,4}, v_2 \sim B_{2,5}, v_3 \sim B_{3,6} \), and \( v_4 \sim B_{5,7} \). The leaf vertices \( u_5 \) and \( u_7 \), and the internal vertex \( v_4 \sim B_{5,7} \) are distinguished. The distinguished tree \( \tau_2 \) is drawn with thick edges.

6. The distinguished tree graph

In this section, we further refine the combinatorial organization of terms corresponding to the distinguished tree. We note that regular trees can be treated in a similar manner, with obvious modifications. Let \( \tau_j \) denote the distinguished tree graph. We assume that it contains \( m_j \) internal vertices \( (v_{j,\alpha})_{\alpha=1}^{m_j} \), and \( m_j + 1 \) leaf vertices \( (u_{j,\alpha})_{\alpha=1}^{m_j+1} \). We recall that the internal vertices are enumerated with \( \alpha \in \{1, \ldots, m_j\} \), where \( \alpha = m_j \) is the distinguished vertex, and that \( \alpha \) corresponds to the interaction operator \( B_{\sigma_j(\alpha+1),\alpha+1} \) (notice the shift by 1 in the index) in (6.3). For notational simplicity, we will from here on label leaf vertices with \( \alpha \in \{m_j + 1, \ldots, 2m_j + 2\} \) (corresponding to \( u_{j,\alpha-m_j} \)), and will often refer to the vertex \( v_{j,\alpha} \) by its label \( \alpha \).
To determine the contribution to \( \{4.26\} \) corresponding to \( \tau_j \), we use the commutativity relation \( \{1.17\} \), and straightforwardly find that
\[
J_j^1(\sigma_j; t, t_{\ell_1}, \ldots, t_{\ell_{m_j}}; x_j; x_j')
= U^{(1)}(t - t_{\ell_1}) \cdots U^{(1)}(t_{\ell_1} - t_{\ell_2}) B_{\sigma_j(2)} \cdots
\cdots B_{\sigma_j(\alpha)} U^{(\alpha)}(t_{\ell_{\alpha-1}} - t_{\ell_{\alpha-1}+1}) \cdots U^{(\alpha)}(t_{\ell_{\alpha-1}} - t_{\ell_{\alpha}}) B_{\sigma_j(\alpha+1), \alpha+1} \cdots
\cdots U^{(m_j)}(t_{\ell_{m_j}} - t_{\ell_{m_j}+1}) B_{\sigma_j(m_j+1), m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(m_j+1)}.
\] (6.1)

Here, the interaction operators \( B_{\sigma_j(\alpha), \alpha} \) are adapted to the 1-particle kernel \( J_j^1 \), and \((\sigma_j)_{j=1}^{m_j}\) is an “internal” labeling of the interaction operators that preserves the structure of \( \tau_j \). In this sense, \( \sigma_j \) corresponds to the restriction of \( \sigma \) to the tree \( \tau_j \). Clearly, \( \sigma_j(2) = 1 \).

Between any two consecutive interaction operators \( B_{\sigma_j(\alpha), \alpha} \) and \( B_{\sigma_j(\alpha+1), \alpha+1} \), with \( \alpha < m_j \), there is a composition of \( \ell_{j, \alpha} - \ell_{j, \alpha-1} \) free propagators at consecutive time steps, so that
\[
U^{(\alpha)}(t_{\ell_{\alpha-1}} - t_{\ell_{\alpha-1}+1}) \cdots U^{(\alpha)}(t_{\ell_{\alpha-1}} - t_{\ell_{\alpha}}) = U^{(\alpha)}(t_{\ell_{\alpha-1}} - t_{\ell_{\alpha}}),
\] (6.2)
due to the group property of the free propagators. Hence, \( (6.1) \) reduces to
\[
J_j^1(\sigma_j; t, t_{\ell_1}, \ldots, t_{\ell_{m_j}}) = U^{(1)}(t - t_{\ell_1}) B_{1,2} \cdots
\cdots B_{\sigma_j(\alpha), \alpha} U^{(\alpha)}(t_{\ell_{\alpha-1}} - t_{\ell_{\alpha}}) B_{\sigma_j(\alpha+1), \alpha+1} \cdots
\cdots U^{(m_j)}(t_{\ell_{m_j}} - t_{\ell_{m_j}+1}) B_{\sigma_j(m_j+1), m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(m_j+1)}
\] (6.3)
where \( \ell_{j, m_j} = r \). We observe that on the last line, there is no free propagator in front of \( (|\phi\rangle\langle\phi|)^{\otimes(m_j+1)} \) (since \( \tau_j \) is distinguished). As a consequence, the Strichartz estimate cannot be applied in the last step. Resolving this issue is the main task of the construction presented in the sequel.

Our goal is to bound
\[
\int_{[0,T]^{m_j-1}} dt_{\ell_1,1} \cdots dt_{\ell_{m_j-1},1} \text{Tr}\(| J_j^1(\sigma_j; t, t_{\ell_1}, \ldots, t_{\ell_{m_j}}) |).
\] (6.4)

We will employ a recursion that takes into account the structure of interactions and free evolutions occurring between interactions. We will explain the strategy based on an example in the next section.

6.1. Example calculation for a distinguished tree. We consider an example of a distinguished tree, which we obtain from setting \( k = 1, r = 3 \) in \( \{4.23\} \) (if \( k = 1 \), there is only one tree, and it is necessarily distinguished). From \( \{4.23\} \) and \( \{4.24\} \), we have
\[
\gamma^{(1)}(t) = (i\lambda)^3 \sum_{\sigma \in N_{1,3}} \int_{D(\sigma, t)} dt_1 dt_2 dt_3 \int d\tilde{\mu}_3(\phi) U^{(1)}(t - t_1) B_{1,2} U^{(2)}(t_1 - t_2)
B_{\sigma(3), 3} U^{(3)}(t_2 - t_3) B_{\sigma(4), 4}(|\phi\rangle\langle\phi|)^{\otimes4},
\] (6.5)
where $D(\sigma, t) \subseteq [0, t]^3$. For a fixed $\sigma$ (with, say, $\sigma(3) = 2$ and $\sigma(4) = 3$), we consider, as an example, the contribution to the bound \[4.27\] of the form

$$
\int_{[0, T]^3} dt_1 dt_2 dt_3 \int d\mu_{\lambda_{i\lambda}^j}(\phi) \text{Tr}\left( \left| \left( U^{(1)}(t - t_1)B_{1,2}U^{(2)}(t_1 - t_2) B_{2,3}U^{(3)}(t_2 - t_3)B_{3,4}(|\phi\rangle\langle\phi|)^{\otimes 4} \right) \right| \right),
$$

where $t \in [0, T]$, and noting that $|i\lambda| = 1$.

### 6.1.1. Recursive determination of contraction structure

Clearly, $(|\phi\rangle\langle\phi|)^{\otimes 4}$ is a product of 1-particle density matrices. We observe that the interaction operators $B_{i,j}$ preserve the product structure (while changing the explicit expressions for each factor), and contract two factors at a time (the $i$-th and the $j$-th). On all other factors, $B_{i,j}$ acts as the identity. Similarly as in the example of Section \[5.1\] we introduce kernels $\Theta_\alpha$, $\alpha = 1, \ldots, 3$ that account for the contractions performed by $B_{\sigma(\alpha+1), \sigma+1}$, which we write in the normal form

$$
\Theta_\alpha(x, x') = \sum_{\beta_\alpha} c_{\beta_\alpha}^\alpha \chi_{\beta_\alpha}^\alpha(x) \bar{\psi}_{\beta_\alpha}^\alpha(x')
$$

(6.7)

where $\chi_{\beta_\alpha}^\alpha, \bar{\psi}_{\beta_\alpha}^\alpha$ are certain functions that will be recursively determined, and $c_{\beta_\alpha}^\alpha$ are coefficients with values in $\{1, -1\}$.

- **The kernel $\Theta_3$:** We start at the last interaction operator $B_{3,4}$ in \[6.6\]. It acts nontrivially only on the 3-rd and 4-th factor in $(|\phi\rangle\langle\phi|)^{\otimes 4}$,

$$
B_{3,4}(|\phi\rangle\langle\phi|)^{\otimes 4} = (|\phi\rangle\langle\phi|)^{\otimes 2} \otimes \Theta_3.
$$

(6.8)

The kernel $\Theta_3$ is obtained from contracting a two particle density matrix to a one particle density matrix via the interaction operator $B_{1,2}$ (which acts on a two-particle kernel $f(x, y; x', y')$ by $(B_{1,2}f)(x, x') = f(x, x'; x) - f(x, x'; x')$),

$$
\Theta_3(x, x') := B_{1,2} \left( (|\phi\rangle\langle\phi|)^{\otimes 2} \right)(x, x') = \bar{\psi}(x)\tilde{\psi}(x') - \phi(x)\bar{\phi}(x')
$$

(6.9)

where

$$
\bar{\psi} := |\phi|^2\phi.
$$

(6.10)

Here, we have $c_1^3 = 1, c_2^3 = -1, \chi_1^3 = \bar{\psi}, \chi_2^3 = \phi, \psi_1^3 = \phi, \psi_2^3 = \bar{\phi}$.

**Main difficulty:** The main difficulty in estimating \[6.6\] stems from the fact that the term $\bar{\psi} = |\phi|^2\phi$ can only be controlled in $L^2$, where by Sobolev embedding, $||\bar{\psi}||_{L^2} \leq C||\phi||_{H^1}$, which can then be controlled by \[4.18\], see \[6.24\] below. Our objective thus is to apply the triangle inequality to the trace norm inside \[6.6\], and to recursively “propagate” the resulting $L^2$ norm through all intermediate terms until we reach $\psi$, see \[6.15\] below. We remark that if $||\tilde{\psi}||_{H^1}$ could be controlled by $||\phi||_{H^1}$ (which is not the case), a straightforward application of the method of \[31\] would suffice to carry out our analysis.
We now re-interpret $\tilde{\psi}$ in (6.9) as a function that is independent of $\phi$, $\bar{\phi}$. Only at the end of our analysis, we will substitute $\tilde{\psi} := |\phi|^2 \phi$. We call a factor $\chi^2_{\beta_n}$, $\psi^2_{\beta_n}$ in the sum (6.7) distinguished if it is a function of $\tilde{\psi}$. In the first step, it is clear that for every $\beta_3$, only one out of the two factors $\chi^2_{\beta_3}, \psi^2_{\beta_3}$ in the sum (6.9) is distinguished (and in fact equal to $\tilde{\psi}$). The property of being distinguished then propagates from there, i.e., in the next step the distinguished term is the one containing the distinguished term as a factor from the previous step, and so on.

- The kernel $\Theta_2$: Next, we consider the terms contracted by $B_{2,3}$ in (6.6).
  \( B_{2,3}U^{(3)}\sum (t_2 - t_3)\left( (|\phi\rangle\langle\phi|)^{\otimes 2} \otimes \Theta_3 \right) = (U^{(1)}(t_2 - t_3)|\phi\rangle\langle\phi|) \otimes \Theta_2, \) (6.11)
  using (6.8), which defines the kernel
  \[
  \Theta_2(x, x') = B_{1,2}\left( (U^{(1)}(t_2 - t_3)|\phi\rangle\langle\phi|) \otimes (U^{(1)}(t_2 - t_3)\Theta_3) \right)(x, x')
  = (U_{2,3}\phi)(x)(U_{2,3}\phi)(x') \sum_{\beta_3=1}^2 c_{\beta_3}^3 \left[ (U_{2,3}\chi_{\beta_3}^3)(x)(U_{2,3}\psi_{\beta_3}^3)(x) - (U_{2,3}\chi_{\beta_3}^3)(x')(U_{2,3}\psi_{\beta_3}^3)(x') \right]
  =: \sum_{\beta_2=1}^4 c_{\beta_2}^2 \chi_{\beta_2}^2(x)\overline{\psi_{\beta_2}^2}(x'), \] (6.12)
  where
  \[ U_{i,j} := e^{i(t_i - t_j)\Delta}. \] (6.13)
  Since for every $\beta_3$, only one out of the two factors $\chi^3_{\beta_3}, \psi^3_{\beta_3}$ is distinguished, it follows from (6.12) that for every $\beta_2 \in \{1, \ldots, 4\}$, only one out of the two factors $\chi^2_{\beta_2}, \psi^2_{\beta_2}$ is distinguished. The coefficients $c_{\beta_2}^2$ again have values in \{1, -1\}.

- The kernel $\Theta_1$: Finally, we consider the terms contracted by $B_{1,2}$ in (6.6), corresponding to
  \[
  \Theta_1(x, x') = B_{1,2}\left( (U^{(1)}(t_1 - t_2)U^{(1)}(t_2 - t_3)|\phi\rangle\langle\phi|) \otimes (U^{(1)}(t_1 - t_2)\Theta_2) \right)(x, x')
  = (U_{1,3}\phi)(x)(U_{1,3}\phi)(x') \sum_{\beta_2=1}^4 c_{\beta_2}^2 \left[ (U_{1,2}\chi_{\beta_2}^2)(x)(U_{1,2}\psi_{\beta_2}^2)(x) - (U_{1,2}\chi_{\beta_2}^2)(x')(U_{1,2}\psi_{\beta_2}^2)(x') \right]
  =: \sum_{\beta_1=1}^8 c_{\beta_1}^1 \chi_{\beta_1}^1(x)\overline{\psi_{\beta_1}^1}(x'). \] (6.14)
  Again, since for every $\beta_2$, only one out of the two functions $\chi^2_{\beta_2}, \psi^2_{\beta_2}$ is distinguished, it follows that for every $\beta_1 \in \{1, \ldots, 8\}$, only one out of the two functions $\chi^1_{\beta_1}, \psi^1_{\beta_1}$ is distinguished. The coefficients $c_{\beta_1}^1$ again have values in \{1, -1\}.
6.1.2. Recursive bounds. We may now return to (6.6), and perform the following recursive bounds with respect to time integration.

- Integral in $t_1$. Applying Cauchy-Schwarz with respect to the integral in $t_1$ and the triangle inequality for the trace norm, we obtain that

$$
[6.6] = \int_{[0,T]^3} dt_1 dt_2 dt_3 \int d\mu_{\epsilon_3}^{(i)}(\phi) \text{Tr}\left( \left| U^{(1)}(t - t_1)\Theta_1 \right| \right)
$$

$$
\leq \sum_{\beta_1 = 1}^{8} T^{1/2} \int_{[0,T]^2} dt_2 dt_3 \int d\mu_{\epsilon_3}^{(i)}(\phi) \left\| \chi^{1}_{\beta_1} \right\|_{L^2_{\tilde{L}}} \left\| \psi^{1}_{\beta_1} \right\|_{L^2_{\tilde{L}}} \left\| U^{(1,2)}_{1,11} \right\|_{L^2_{\tilde{L}}} (6.15)
$$

using that $|c^{1}_{\beta_1}| = 1$. From (6.14), we see that given $\beta_1 \in \{1, \ldots, 8\}$, there exists $\beta_2$ such that

$$
\chi^{1}_{\beta_1}(x) = (U_{1,3}\phi)(x)
$$

$$
\psi^{1}_{\beta_1}(x) = (U_{1,3}\phi)(x)(U_{1,2}\chi^{3}_{\beta_2})(x)(U_{1,2}\psi^{3}_{\beta_2})(x)
$$

(6.16)

(or with a cubic expressions for $\chi^{1}_{\beta_1}$ and a linear expression for $\psi^{1}_{\beta_1}$). Therefore,

$$
\left\| (U_{1,3}\phi)(x)(U_{1,2}\chi^{3}_{\beta_2})(x)(U_{1,2}\psi^{3}_{\beta_2})(x) \right\|_{L^2_{t_1}([0,T])\times L^2} = \|\phi\|_{L^2} \left\| \chi^{3}_{\beta_1} \right\|_{L^2_{\tilde{T}}} \left\| \psi^{3}_{\beta_1} \right\|_{L^2_{\tilde{T}}} \left\| U_{1,3} \right\|_{L^2_{\tilde{T}}} (6.17)
$$

using that $U_{1,3}$ is unitary, and that $\phi$ does not depend on $t_1$.

Next, we observe that

$$
\left\| (e^{it\Delta}f_1)(x)(e^{it\Delta}f_2)(x)(e^{it\Delta}f_3)(x) \right\|_{L^2_{t}(\mathbb{R})L^2_{x}(\mathbb{R}^3)} \leq \|e^{it\Delta}f_1\|_{L^\infty_t L^6_x} \|e^{it\Delta}f_2\|_{L^\infty_t L^6_x} \|e^{it\Delta}f_3\|_{L^1_t L^6_x} \leq C \|f_1\|_{H^1} \|f_2\|_{H^1} \|f_3\|_{H^1} \|f_3\|_{L^6_x} (6.18)
$$

using the Hölder inequality, the Sobolev inequality, and the Strichartz estimate $\|e^{it\Delta}f\|_{L^6_t L^6_x} \leq C \|f\|_{L^2}$ for the free Schrödinger evolution. We make the important observation that in (6.18), we can place the $L^2_x$-norm on any of the three functions $f_j$, $j = 1, 2, 3$, and not only on $f_3$. Similarly, if a derivative is included,

$$
\left\| \nabla_x (e^{it\Delta}f_1)(x)(e^{it\Delta}f_2)(x)(e^{it\Delta}f_3)(x) \right\|_{L^2_{t}(\mathbb{R})L^2_{x}(\mathbb{R}^3)} \leq \sum_{j=1}^{3} \|e^{it\Delta}\nabla_x f_j\|_{L^2_{t}L^6_{x}} \prod_{1 \leq i \leq 3, i \neq j} \|e^{it\Delta}f_i\|_{L^\infty_t L^6_x} \leq C \|f_1\|_{H^1} \|f_2\|_{H^1} \|f_3\|_{H^1} \|f_3\|_{H^1} \|f_3\|_{H^1} (6.19)
$$

which, together with (6.18), implies that

$$
\left\| (e^{it\Delta}f_1)(x)(e^{it\Delta}f_2)(x)(e^{it\Delta}f_3)(x) \right\|_{L^2_{t}(\mathbb{R})H^1_{x}(\mathbb{R}^3)} \leq C \prod_{j=1}^{3} \|f_j\|_{H^1_x} (6.20)
$$

Only one of the factors $\chi^{3}_{\beta_2}, \psi^{3}_{\beta_2}$ is distinguished, say for instance $\psi^{3}_{\beta_2}$. We then use (6.18) in such a way that the $L^2_x$-norm is applied to this term. All terms in
By assumption, the factor
Subsequently, we obtain

\[ \mathcal{L}_\chi(x) \leq \sup_{t \in [0,T]} \left( \int_{t}^{\infty} \left| \mathcal{L}_\chi(t) \right| dt \right)^{1/2} \]

where the indices \( \beta_2 \) depend on \( \beta_1 \). Next, we use the defining relation (6.12) for the functions \( \chi_{\beta_2}^2, \psi_{\beta_2}^2 \), and consider the integral in \( t_2 \).

- **Integral in \( t_2 \).** By assumption, the factor \( \psi_{\beta_3}^3 \) is distinguished, while \( \chi_{\beta_3}^3 \) is not. Moreover, one of the functions \( \chi_{\beta_2}^2, \psi_{\beta_2}^2 \) is a linear, while the other one is a cubic expression in the functions after the second equality sign in (6.12) (the distinguished factor could be either). Our goal is to bound the distinguished factor in \( L^2 \). From comparing terms in (6.12), one possible combination is

\[ \chi_{\beta_2}^2(x) = (U_{2,3}\phi)(x) , \quad \psi_{\beta_2}^2(x) = (U_{2,3}\phi)(x)(U_{2,3}x_{\beta_3})(x)(U_{2,3}x_{\beta_3})(x), \]

that is, the distinguished factor \( \psi_{\beta_2}^2 \) is a cubic expression. We apply Cauchy-Schwarz in the \( t_2 \)-integral in such a way that the \( L^2 \)-norm falls on the cubic term. (If, on the other hand, \( \chi_{\beta_3}^3 \) is the cubic term, we use Cauchy-Schwarz in \( t_2 \) to get \( \|\psi_{\beta_2}^2\|_{L^2_t L^2_x} \) and \( \|\chi_{\beta_3}^3\|_{L^2_t L^2_x} \) \leq C \|\phi\|_{H^1}^3 \) from (6.20).) We then get

\[ \mathcal{L}_\chi(x) \leq CT \sum_{\beta_1=1}^{8} \int_{[0,T]} dt_3 \int d\mu_3^{(i)}(\phi) \|\phi\|^2_{H^1} \|\chi_{\beta_2}^2\|_{L^2_t L^2_x} \|\psi_{\beta_2}^2\|_{L^2_t L^2_x} \]

where only one of the three factors inside the norm on the last line is distinguished. We may assume it is \( \psi_{\beta_3}^3 \). By comparing terms in (6.9), we then find that \( \psi_{\beta_3}^3 = \psi \), and \( \chi_{\beta_3}^3 = \phi \). We then apply (6.18) again, and use the \( L^2 \)-bound for \( \psi_{\beta_3}^3 = \psi \). At this point, we substitute \( \psi = |\phi|^2 \phi \).

- **Using de Finetti for the last step.** Subsequently, we obtain

\[ \mathcal{L}_\chi(x) \leq CT \sum_{\beta_1=1}^{8} \int_{[0,T]} dt_3 \int d\mu_3^{(i)}(\phi) \|\phi\|^2_{H^1} \|\psi\|_{L^2} \]

where we used \( \|\psi\|_{L^2} \leq C \|\phi\|_{H^1} \) from Sobolev embedding, and the bound (4.18) related to the de Finetti theorem, which is uniform in \( t_3 \). This is the desired estimate in our example calculation.

The strategy presented in this example can be applied in the general case.
Figure 2. An example of a distinguished tree $\tau_j$ with $m_j = 7$. The number next to an internal vertex corresponds to its label $\alpha \in \{1, \ldots, 7\}$, ordered increasingly from left to right. The vertex with label $\alpha$ corresponds to $B_{\sigma_j(\alpha+1), \alpha+1}$ (note the shift by 1 in the index). The time variable attached to it is $t_{\ell_j, \alpha}$. The leftmost vertex is the root vertex, while the rightmost vertices are leaf vertices labeled with $\alpha \in \{8, 15\}$, corresponding to $u_{j, \alpha - 7}$. The leaf vertices 8 to 13 are regular, while the leaf vertices 14, 15 are distinguished. The distinguished internal vertex has label 7. Here, for example, $\kappa_- (1) = 2$, $\kappa_+ (1) = 4$, $\kappa_+ (\kappa_+ (\kappa_+ (1))) = 7$, $\kappa_- (2) = 3$.

This tree corresponds to the term

$$U^{(1)}_{0,1}B_{1,2}U^{(2)}_{1,2}B_{1,3}U^{(3)}_{2,3}B_{1,4}U^{(4)}_{3,4}B_{2,5}U^{(5)}_{4,5}B_{5,6}U^{(6)}_{5,6}B_{5,7}U^{(7)}_{6,7}B_{7,8}(\langle \phi \rangle \langle \phi \rangle) \otimes 8 \quad (6.25)$$

(in upper echelon form) where $U^{(j)}_{i,j} = U^{(j)}(t_i - t_j)$ and $t_0 = t$.

6.2. Recursive definition of kernels at vertices. As in the above example, we recursively assign a kernel $\Theta_\alpha$ to each vertex $\alpha$. As the root of this induction, we associate the kernel

$$\Theta_\alpha(x; x') := \phi(x)\phi(x') \quad (6.26)$$

to the leaf vertex with label $\alpha \in \{m_j + 1, \ldots, 2m_j + 2\}$ (corresponding to $u_{j, \alpha - m_j}$).

In the first recursion step, we determine $\Theta_{m_j}$ at the distinguished vertex $\alpha = m_j$ from the term on the last line of (6.24), given by

$$B_{\sigma_j(m_j+1), m_j+1} (|\phi \rangle \langle \phi |) \otimes (m_j+1) = (|\phi \rangle \langle \phi |) \otimes (\sigma(m_j+1)-1) \otimes \Theta_{m_j}$$

$$\otimes (|\phi \rangle \langle \phi |) \otimes (m_j+1-\sigma(m_j+1)-1) \quad (6.27)$$

where

$$\Theta_{m_j}(x; x') := \bar{\psi}(x)\overline{\phi(x')} - \phi(x)\bar{\psi}(x') \quad (6.28)$$
with $\tilde{\psi}$ as in (6.10). It is obtained from contracting the two copies of $|\phi\rangle\langle\phi|$ at the two leaf vertices $\kappa_-(m_j)$, $\kappa_+(m_j)$ which have $m_j$ as their parent vertex.

For the induction step, we let $\alpha \in \{1, \ldots, m_j - 1\}$ label a regular internal vertex, and assume that the kernels $\Theta_{\kappa'}$ have been determined for all $\alpha' > \alpha$. Let $\kappa_-(\alpha)$, $\kappa_+(\alpha)$ label the two child vertices (of internal or leaf type) of $\alpha$,

$$
\sigma_j(\alpha) = \sigma_j(\kappa_-(\alpha)) \quad \alpha = \sigma_j(\kappa_+(\alpha)).
$$

Then, by induction assumption, $\Theta_{\kappa_-(\alpha)}$, $\Theta_{\kappa_+(\alpha)}$ are given, and we define

$$
\Theta_{\alpha}(x; x') = B_{1,2} \left( \left( U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)})\Theta_{\kappa_-(\alpha)} \right) \otimes \left( U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)} \right) \right)(x; x')
$$

then, by induction assumption, $\Theta_{\kappa_-(\alpha)}$, $\Theta_{\kappa_+(\alpha)}$ are given, and we define

$$
\Theta_{\alpha}(x; x') = B_{1,2} \left( \left( U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)})\Theta_{\kappa_-(\alpha)} \right) \otimes \left( U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)} \right) \right)(x; x')
$$

Clearly, if $\kappa_+(\alpha)$ corresponds to a regular leaf vertex, then

$$
\Theta_{\kappa_+(\alpha)}(x; x') = \phi(x)\bar{\phi}(x'),
$$

and $t_{\ell,\kappa_+(\alpha)} = t_r$. If $\kappa_+(\alpha) = m_j$ is the distinguished vertex, we use (6.28).

We iterate this procedure until we obtain the kernel $\Theta_1$ at $\alpha = 1$, which is the unique child vertex of the root vertex.

### 6.3. Factorization structure of kernels

We will now determine the structure of $\Theta_{\alpha}$.

**Lemma 6.1.** Let $\alpha \in \{1, \ldots, m_j\}$. Then, every kernel $\Theta_{\alpha}$ can be written as a sum of differences of factorized kernels,

$$
\Theta_{\alpha}(x; x') = \sum_{\beta_\alpha} c_{\beta_\alpha}^\alpha \chi_{\beta_\alpha}(x)\bar{\psi}_{\beta_\alpha}(x')
$$

with at most $2^{m_j-\alpha}$ nonzero coefficients $c_{\beta_\alpha}^\alpha \in \{1, -1\}$.

**Proof.** The kernels at the leaf vertices (6.26) have the form (6.32). If $\alpha = m_j$ is the distinguished vertex, $\Theta_{m_j}$ is given by (6.28), and evidently has the form (6.32). For the induction step, let us assume that given $\alpha \in \{1, \ldots, m_j\}$, the kernels $\Theta_{\alpha'}$ have the form (6.32) for all $\alpha' > \alpha$, thus in particular $\Theta_{\kappa_+(\alpha)}$, $\Theta_{\kappa_-(\alpha)}$ have this form. Then, from (6.30), we find that

$$
\Theta_{\alpha}(x; x') = \sum_{\beta_{\kappa_-(\alpha)}; \beta_{\kappa_+(\alpha)}} c_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)} c_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)} (U_{\alpha;\kappa_-(\alpha)}\chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}(x)(U_{\alpha;\kappa_-(\alpha)}\bar{\psi}_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}(x'))
$$

$$
- (U_{\alpha;\kappa_+(\alpha)}\chi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}(x)(U_{\alpha;\kappa_+(\alpha)}\bar{\psi}_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}(x'))
$$

where for brevity, we write

$$
U_{\alpha;\alpha'} := e^{i(t_{\ell,\alpha} - t_{\ell,\alpha'})\Delta_x}, \quad \alpha, \alpha' \in \{0, 1, \ldots, m_j\},
$$

(6.34)
(with \(t_0 := t\) for free one-particle propagators. We write the sum on the right hand side in an arbitrary but fixed order, and use the individual terms as definitions for the terms \(c^\alpha_{\beta a}, \chi^\alpha_{\beta a}, \psi^\alpha_{\beta a}\) in

\[
\Theta_\alpha(x; x') = \sum_{\beta a} c^\alpha_{\beta a} \chi^\alpha_{\beta a}(x) \overline{\psi^\alpha_{\beta a}}(x'),
\]

which is of the form (6.32). This iteration terminates when we reach \(\alpha = 1\). □

In particular, we have that

\[
J_j^1(\alpha; t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,n_j}}) = U^{(1)}(t - t_{\ell_{j,1}})\Theta_1.
\]

For convenience, we will notationally suppress the dependence of the functions \(\chi^\alpha_{\beta a}, \psi^\alpha_{\beta a}\) on the time variables \(t_{\ell_{a'}}\), but we note that they do not depend on any \(t_{\ell_{a'}}\) with \(a' < a\).

**Definition 6.2.** From here on, we re-interpret \(\tilde{\psi}\) in \(\Theta_{m_j}\) (see (6.28)) as a function which is independent of \(\phi, \tilde{\phi}\). Only at the end of our analysis, we will substitute \(\tilde{\psi} := |\phi|^2\phi\). For each \(\alpha = 1, \ldots, m_j\), we call a factor \(\chi^\alpha_{\beta a}, \psi^\alpha_{\beta a}\) in the expansion (6.32) distinguished if it is a function of \(\tilde{\psi}\).

Next, we derive recursive bounds on the functions \(\chi^\alpha_{\beta a}, \psi^\alpha_{\beta a}\).

### 6.4. Key properties of the kernels \(\Theta_\alpha\). We make the following key observations which will be crucial for the next steps of our proof:

- The only dependence of \(\Theta_\alpha\) on the time variable \(t_{\ell_{j,a}}\) is via the propagators
  \[
  U_{\alpha;\kappa_{\pm}(\alpha)} = e^{i(t_{\ell_{j,a}} - t_{\ell_{j,\kappa_{\pm}(\alpha)})} \Delta
  \]
  appearing on the right hand side of (6.33). The kernels \(\Theta_{\kappa_{\pm}(\alpha)}\) at the two child vertices \(\kappa_{\pm}(\alpha)\) of \(\alpha\) do not depend on \(t_{\ell_{j,a}}\). This will be crucial for the application of Strichartz estimates below.

- The product \(\chi^\alpha_{\beta a}(x)\overline{\psi^\alpha_{\beta a}}(x')\) in (6.35) either has the form
  \[
  \chi^\alpha_{\beta a}(x)\overline{\psi^\alpha_{\beta a}}(x') = (U_{\alpha;\kappa_-(\alpha)}\chi^\alpha_{\beta a_-(\alpha)})(x)(U_{\alpha;\kappa_-(\alpha)}\psi^\alpha_{\beta a_-(\alpha)})(x')
  \]
  or
  \[
  \chi^\alpha_{\beta a}(x)\overline{\psi^\alpha_{\beta a}}(x') = (U_{\alpha;\kappa_+(\alpha)}\chi^\alpha_{\beta a_+(\alpha)})(x)(U_{\alpha;\kappa_+(\alpha)}\psi^\alpha_{\beta a_+(\alpha)})(x')
  \]
  for some values of \(\beta_{\kappa_-(\alpha)}, \beta_{\kappa_+(\alpha)}\) that depend on \(\beta_{\alpha}\). Comparing the left and right hand sides, the function \(\chi^\alpha_{\beta a}\) either has the cubic form
  \[
  \chi^\alpha_{\beta a}(x) = (U_{\alpha;\kappa_-(\alpha)}\chi^\alpha_{\beta a_-(\alpha)})(x)
  \]
  or
  \[
  \chi^\alpha_{\beta a}(x) = (U_{\alpha;\kappa_+(\alpha)}\chi^\alpha_{\beta a_+(\alpha)})(x)\]
  in (6.38), (6.39).

\[
\chi^\alpha_{\beta a}(x) = (U_{\alpha;\kappa_-(\alpha)}\chi^\alpha_{\beta a_-(\alpha)})(x)\]

\[
\chi^\alpha_{\beta a}(x) = (U_{\alpha;\kappa_+(\alpha)}\chi^\alpha_{\beta a_+(\alpha)})(x)
\]
or the linear form
\[ \chi_{\beta_{\alpha}}^{\alpha}(x) = (U_{\alpha;\kappa^{-}(-\alpha)}\chi_{\beta_{\alpha}^{-}(-\alpha)})(x). \] (6.41)
Accordingly, \( \psi_{\beta_{\alpha}}^{\alpha} \) respectively has either linear or cubic form. It is important that the product \( \chi_{\beta_{\alpha}}^{\alpha} \psi_{\beta_{\alpha}}^{\alpha} \) is always of quartic form (6.38) or (6.39). This fact will again be crucial for the application of Strichartz estimates below.

• In the product on the right hand side of (6.38), respectively (6.39), at most one of the four factors is distinguished (see Definition (6.2)). This follows straightforwardly from an induction along decreasing values of \( \alpha \), using the fact that the statement is true for all regular leaf vertices, and for the distinguished vertex (6.28).

We may therefore make the following assumption, which leads to notational simplifications, but to no loss of generality.

**Hypothesis 6.1.** In all that follows, we assume for notational convenience that only the functions \( \psi_{\beta_{\alpha}}^{1} \), and recursively, \( (\psi_{\beta_{\alpha}}^{Q_{n}(1)})_{q=1}^{Q} \), are distinguished (i.e., are a function of \( \tilde{\psi} \) in (6.28)). Here,
\[ \kappa_{q}^{+}(1) := \kappa_{+}(\kappa_{+}(\cdots(\kappa_{+}(1))\cdots)), \] (6.42)
is the \( q \)-th iterate of the index \( \alpha = 1 \) under \( \kappa_{+} \), and \( Q \) is the number of edges linking \( \alpha = 1 \) to the distinguished vertex with label \( \alpha = m_{j} \). This is one special case, but all cases can be treated in the same way.

7. **Recursive \( L^{2} \)- and \( H^{1} \)-bounds for the distinguished tree**

From here on, we abbreviate the notation by writing \( t_{\alpha} \) for \( t_{j_{v},\alpha} \), and by referring to the vertex \( v_{j_{v},\alpha} \) by its label \( \alpha \). Also, we will say that the time variable \( t_{\alpha} \) is attached to the vertex \( \alpha \).

We have that
\[
\int_{[0,T)^{m_{j}-1}} dt_{1} \cdots dt_{m_{j}-1} \text{Tr} \left( \left| J^1_{j}(\sigma_{j}; t, t_{1}, \ldots, t_{m_{j}}) \right| \right) \\
= \int_{[0,T)^{m_{j}-1}} dt_{1} \cdots dt_{m_{j}-1} \text{Tr} \left( \left| U^{(1)}(t - t_{1})\Theta_{1} \right| \right) \\
\leq \sum_{\beta_{1}} \int_{[0,T)^{m_{j}-1}} dt_{1} \cdots dt_{m_{j}-1} \| \psi_{\beta_{1}}^{1} \|_{L^{2}} \| \chi_{\beta_{1}}^{1} \|_{L^{2}}. \] (7.1)

We will estimate the last term on the right hand side based on the recursion formula (6.33), using recursive bounds adapted to a hierarchy of subtrees of \( \tau_{j} \).

Our main goal is to propagate the \( L^{2} \)-norm in (7.1) along edges of \( \tau_{j} \) that connect the vertex \( \alpha = 1 \) to the distinguished vertex \( \alpha = m_{j} \), in order to obtain a bound
\[ \| \tilde{\psi} \|_{L^{2}} = \| \phi^{2} \phi \|_{L^{2}} \leq C\| \phi \|_{H^{1}}, \] (7.2)
which can be controlled by the growth condition (4.18).
7.1. **Recursive bounds.** We let $\tau_{j,\alpha}$ denote the subtree of $\tau_j$ with root at the vertex labeled by $\alpha$. Moreover, we denote by

$$
\int \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \equiv \int_{[0,T]^{d_\alpha}} \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right]
$$

(7.3)

integration with respect to all time variables attached to the internal and root vertices of the subtree $\tau_{j,\alpha}$ with root at $\alpha$. Here, $d_\alpha$ denotes the total number of internal and root vertices of $\tau_{j,\alpha}$.

**Lemma 7.1.** Let $\kappa_-(\alpha)$ and $\kappa_+\alpha$) label the two child vertices of the vertex labeled by $\alpha$. Assume that either (6.38) or (6.39) is given. Then, the following recursive bounds hold:

- **Bound on $L^2$-level**

$$
\int \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \| \psi_{\beta_{\alpha}}^\alpha \|_{L^2} \| \chi_{\beta_{\alpha}}^\alpha \|_{H^1} \leq C T^{1/2} \int \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \| \psi_{\beta_{\alpha}}^\alpha \|_{H^1} \| \chi_{\beta_{\alpha}}^\alpha \|_{H^1}
$$

(7.4)

- **Bound on $H^1$-level**

$$
\int \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \| \psi_{\beta_{\alpha}}^\alpha \|_{H^1} \| \chi_{\beta_{\alpha}}^\alpha \|_{H^1} \leq C T^{1/2} \int \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \| \psi_{\beta_{\alpha}}^\alpha \|_{H^1} \| \chi_{\beta_{\alpha}}^\alpha \|_{H^1}
$$

(7.5)

**Proof.** This can be inferred from the bounds (6.18) and (6.20) as follows:

- **Bound on $L^2$-level.** By applying (6.18) to (6.40) and (6.41) with respect to the time variable $t_\alpha$, and recalling that

$$
U_{\alpha;\kappa_-(\alpha)} = e^{i(t_\alpha - t_{\kappa_-(\alpha)})\Delta},
$$

(7.6)

we obtain

$$
\int_{[0,T]^{d_\alpha}} \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \| \psi_{\beta_{\alpha}}^\alpha \|_{L^2} \| \chi_{\beta_{\alpha}}^\alpha \|_{H^1}
\leq\ C T^{1/2} \int_{[0,T]^{d_\alpha-1}} \left[ \prod_{\alpha' \in \tau_{j,\alpha'} \cup \tau_{j,\alpha}} dt_{\alpha'} \right] \| e^{-i\Delta t_{\kappa_-(\alpha)}} \chi_{\beta_{\kappa_-(\alpha)}}^\alpha \|_{H^1}
\| e^{-i\Delta t_{\kappa_+(\alpha)}} \chi_{\beta_{\kappa_+(\alpha)}}^\alpha \|_{H^1}
\| e^{-i\Delta t_{\kappa_+(\alpha)}} \chi_{\beta_{\kappa_+(\alpha)}}^\alpha \|_{L^2}
\leq\ C T^{1/2} \int_{[0,T]^{d_\alpha-1}} \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \| \chi_{\beta_{\kappa_-(\alpha)}}^\alpha \|_{H^1}
\| \chi_{\beta_{\kappa_+(\alpha)}}^\alpha \|_{H^1}
\| \chi_{\beta_{\kappa_+(\alpha)}}^\alpha \|_{L^2}
\int_{[0,T]^{d_\alpha-1}} \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \| \chi_{\beta_{\kappa_-(\alpha)}}^\alpha \|_{H^1}
\| \chi_{\beta_{\kappa_+(\alpha)}}^\alpha \|_{H^1}
\| \chi_{\beta_{\kappa_+(\alpha)}}^\alpha \|_{L^2}
$$

(7.7)

Here, we first used Cauchy-Schwarz in the $t_\alpha$-integral. In the last step, we used that $\psi_{\beta_{\alpha}}^\alpha, \chi_{\beta_{\alpha}}^\alpha$ depend only on the time variables $t_{\alpha'}$ attached to the vertices of the
subtree $\tau_j, \alpha$ rooted at the vertex $\alpha$, for every $\alpha \in \{1, \ldots, m_j - 1\}$. Moreover, we used that $e^{-it\kappa^\pm(\alpha)\Delta}$ are unitary in $L^2$ and $H^1$.

- **Bound on $H^1$-level.** Using \textbf{(6.20)}, we obtain

$$
\int_{[0,T]^{d_{\alpha_0}}} \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \| \psi_{\beta_{\alpha}}^0 \|_{H^1} \| \chi_{\beta_0}^0 \|_{H^1} 
\leq CT^{1/2} \int_{[0,T]^{d_{\alpha_0}}} \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \| e^{-it\kappa^-(\alpha)\Delta} \chi_{\beta_{\alpha}}^- \|_{H^1} 
\leq CT^{1/2} \int_{[0,T]^{d_{\alpha_0}}} \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \| \psi_{\beta_{\alpha}}^0 \|_{H^1} \| e^{-it\kappa^+(\alpha)\Delta} \psi_{\beta_{\alpha}}^+ \|_{H^1} 
= CT^{1/2} \int_{[0,T]^{d_{\alpha_0}}} \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \| \chi_{\beta_{\alpha}}^- \|_{H^1} \| \psi_{\beta_{\alpha}}^+ \|_{H^1} \| \psi_{\beta_{\alpha}}^+ \|_{H^1},
$$

(7.8)

by proceeding as above for the bounds on the $L^2$-level.

\[ \square \]

8. **Concluding the proof**

Using Lemma \textbf{7.1}, we can now prove the following main estimates for the distinguished tree in Proposition \textbf{8.1} and for regular trees in Proposition \textbf{8.2}.

**Proposition 8.1.** Assume that $\tau_j$ is the distinguished tree. Then, the bound

$$
\int_{[0,T]^{m_j-1}} dt_1 \cdots dt_{m_j-1} \left| \operatorname{Tr} \left( J_j^1(\sigma_j; t, t_1, \ldots, t_{m_j}) \right) \right|
\leq 2^{m_j} C^{m_j} T^{(m_j-1)/2} \| \phi \|_{H^1}^{m_j+1}
$$

(8.1)

holds.

**Proof.** To begin with,

$$
\int_{[0,T]^{m_j-1}} dt_1 \cdots dt_{m_j-1} \left| \operatorname{Tr} \left( J_j^1(\sigma_j; t, t_1, \ldots, t_{m_j}) \right) \right|
= \int_{[0,T]^{m_j-1}} dt_1 \cdots dt_{m_j-1} \left| \operatorname{Tr} \left( U(1)(t - t_1)\Theta_1 \right) \right|
\leq \sum_{j_1} \int_{[0,T]^{m_j-1}} dt_1 \cdots dt_{m_j-1} \| \psi_{\beta_1}^1 \|_{L^2} \| \chi_{\beta_1}^1 \|_{L^2}
\leq \sum_{j_1} CT^{1/2} \int \left[ \prod_{\alpha' \in \tau_j, \alpha_{j_1}} dt_{\alpha'} \right] \| \psi_{\beta_{\alpha}}^{-1} \|_{H^1} \| \chi_{\beta_{\alpha}}^{-1} \|_{H^1} \| \psi_{\beta_{\alpha}}^+ \|_{L^2} \| \chi_{\beta_{\alpha}}^+ \|_{L^2},
$$

(8.2)

where we first used \textbf{(6.36)}, then \textbf{(6.35)}, and subsequently \textbf{(7.3)} with respect to the integral in $t_1$, at the vertex $\alpha = 1$. Then, we used \textbf{(6.33)}, and the fact that
\[|c_{\kappa_j}^{(1)}| = 1. \text{ By Hypothesis 6.1, } \psi_{\beta_{\kappa_j}}^{\kappa_j} \text{ is the only function on the last two lines that is distinguished, which is why it is the only term bounded in } L^2.\]

We first bound the integral (8.2). To this end, we iterate the bound (7.5) on the \(H^1\)-level until we reach all leaf vertices of the subtree \(\tau_{j,\kappa_-(1)}\). It follows from Hypothesis 6.1 that \(\tau_{j,\kappa_-(1)}\) does not contain the distinguished vertex, therefore all leaf vertices of \(\tau_{j,\kappa_-(1)}\) are regular. Then, we find that

\[
\int \prod_{\alpha' \in \tau_{j,\kappa_-(1)}} dt_{\alpha'} \|\psi_{\beta_{\kappa_j}}^{\kappa_j}\|_{H^1} \|\chi_{\beta_{\kappa_j}}^{\kappa_j}\|_{H^1} \leq CT^{d_{\kappa_j} - (1)/2} \|\phi\|_{H^1}^{2b_{\kappa_j} - (1)} \quad (8.4)
\]

where \(b_{\kappa_j}\) is the number of regular leaf vertices of the subtree \(\tau_{j,\alpha}\) rooted at \(\alpha\), and \(d_{\kappa_j}\) is the number of internal vertices of the subtree \(\tau_{j,\alpha}\).

Next, we bound the integral (8.3). To this end, we iterate both the bound (7.4) on the \(L^2\)-level, and the bound (7.5) on the \(H^1\)-level until we reach all leaf vertices of the subtree \(\tau_{j,\kappa_-(1)}\), including the distinguished vertex \(v_{j,m_j}\). The \(L^2\) norm is in every step applied to the term \(\psi_{\beta_{\kappa_j}}^{\kappa_j}\), which is the only function in (6.38), respectively (6.39), which is distinguished, due to Hypothesis 6.1. The iteration terminates when all regular leaf vertices, and the distinguished vertex are reached. We then obtain that

\[
\int \prod_{\alpha' \in \tau_{j,\kappa_-(1)}} dt_{\alpha'} \|\psi_{\beta_{\kappa_j}}^{\kappa_j}\|_{L^2} \|\chi_{\beta_{\kappa_j}}^{\kappa_j}\|_{H^1} \leq CT^{(d_{\kappa_j} - (1)/2)} \|\phi\|_{H^1}^{b_{\kappa_j} - (1) + 3} \quad (8.5)
\]

where the \(L^2\)-norm has been moved to the distinguished vertex, hence the factor \(\|\psi\|_{L^2}\). At this point, we substituted \(\psi := |\phi|^2 \phi\), and used the Sobolev embedding.

Combining the bounds (8.4) and (8.5), we obtain (8.1), where all leaves contribute a factor \(\|\phi\|^2 H^1\). The factor \(2^m_j\) bounds the number of terms in the sum over \(\beta_{\kappa_-(1)}, \beta_{\kappa_j}^{\kappa_j}\) in (8.2).

Similarly, we find for regular trees:

**Proposition 8.2.** Assume that \(\tau_j\) is a regular tree. Then, the bound

\[
\int_{[0,T)^m_j} dt_1 \cdots dt_m \text{Tr} \left( \left| J_j^1(\sigma_j; t, t_1, \ldots, t_m) \right| \right) \leq 2^m_j C^{m_j} T^m_j / 2 \|\phi\|_{H^1}^{m_j + 1} \quad (8.6)
\]

holds.

**Proof.** For a regular tree, we have

\[
J_j^1(\sigma_j; t, t_1, \ldots, t_m) = U(t - t_1) B_{\sigma_j(2), 2} \cdots U(t - t_{m-1}) B_{\sigma_j(m_j, m_j + 1)} U(t_{m_j} + 1) \langle |\phi| \langle \phi \rangle \rangle^{(m_j + 1)}.
\]

}\[30\]
The key difference between (8.7) and the corresponding expression (6.3) for the distinguished tree is the presence of the free propagator $U^{(m_j+1)}(t_m)$ on the last line. The proof is immediately obtained from the proof of Proposition 8.1, by using
\[
\int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \text{Tr} \left( J^1_j (\sigma_j; t, t_1, \ldots, t_{m_j}) \right)
\]
\[
= \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \text{Tr} \left( | U^{(1)}(t-t_1) \Theta_1 | \right)
\]
\[
\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \| \psi_\beta_1 \|_{L^2} \| \chi_{\beta_1} \|_{L^2}
\]
\[
\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \| \psi_\beta_1 \|_{H^1} \| \chi_{\beta_1} \|_{H^1},
\]
and by iterating the bound (7.5) on the $H^1$-level until we reach all leaf vertices of $\tau_j$. Because all leaves of $\tau_j$ are regular, no $L^2$-level bound is necessary. \Box

Going back to (4.27), we find from
\[
J^k(\sigma; t, t_1, \ldots, t_r; x_k; x'_k) = \prod_{j=1}^k J^1_j (\sigma_j; t, t_{\ell_j}, \ldots, t_{\ell_j, m_j}; x_j; x'_j)
\]
that
\[
\int_{[0,t]^{r-1}} dt_1 \cdots dt_{r-1} \text{Tr} \left( | J^k (\sigma; t, t_1, \ldots, t_r) | \right)
\]
\[
= \int_{[0,t]^{r-1}} dt_1 \cdots dt_{r-1} \prod_{j=1}^k \text{Tr} \left( | J^1_j (\sigma_j; t, t_{\ell_j}, \ldots, t_{\ell_j, m_j}) | \right)
\]
\[
\leq 2^r C'T^{(r-1)/2} \| \phi \|_{H^1}^{2(k+r)},
\]
by combining the estimates from the $k-1$ regular trees, and from the distinguished tree. The factor $2^r$ is obtained from the product of factors $2^{m_j}$ from all trees $\tau_j$, both regular and distinguished.

Then, we observe that for $t \in [0,T)$,
\[
\text{Tr} \left( | \gamma^{(k)}(t) | \right) \leq (2CT)^{(r-1)/2} \sum_{i=1,2} \int_0^T dt_r \int d\mu^{(i)}_{t_r}(\phi) \| \phi \|_{H^1}^{2(r+k)}
\]
\[
\leq (2CT)^{(r+1)/2} \sup_{t_r \in [0,T)} \sum_{i=1,2} \int d\mu^{(i)}_{t_r}(\phi) \| \phi \|_{H^1}^{2(r+k)}. \tag{8.11}
\]
The growth condition (4.18) implies that this is bounded by
\[
\gamma^{(k)}(t) < 2 M^{2k-2} (2CM^4T)^{(r+1)/2} \rightarrow 0 \quad (r \rightarrow \infty)
\]
for $T < (2CM^4)^{-1}$ (which is in particular uniform in $k$). Since $k$ is fixed and $r$ is arbitrary, we conclude that
\[
\text{Tr} \left( | \gamma^{(k)}(t) | \right) = 0, \quad t \in [0,T),
\]
which implies that $\gamma^{(k)}(t) = 0$ for $t \in [0,T)$, and hence, uniqueness holds.
Moreover, it can be easily checked that

$$\gamma^{(k)}(t) = \int d\mu(\phi) (|S_t(\phi)\rangle \langle S_t(\phi)|)^{\otimes k}, \quad \forall k \in \mathbb{N},$$  \hspace{1cm} (8.14)

is a mild solution of the GP hierarchy in $L^\infty_{t \in [0,T]} \mathfrak{g}^1$ with initial data

$$\gamma^{(k)}(0) = \int d\mu(\phi) (|\phi\rangle \langle \phi|)^{\otimes k}, \quad \forall k \in \mathbb{N}.$$  \hspace{1cm} (8.15)

By uniqueness, it is the only such solution. This proves Theorem 2.3. \hfill \Box

Acknowledgements. We thank Benjamin Schlein, Mathieu Lewin, and an anonymous referee for very helpful remarks. The work of T.C. was supported by NSF grants DMS-1009448 and DMS-1151414 (CAREER). The work of N.P. was supported by NSF grant DMS-1101192. The work of R.S. was supported by NSERC.

References


T. Chen, Department of Mathematics, University of Texas at Austin.
*E-mail address:* tc@math.utexas.edu

C. Hainzl, Fachbereich Mathematik, Universität Tübingen, Germany.
*E-mail address:* christian.hainzl@uni-tuebingen.de

N. Pavlović, Department of Mathematics, University of Texas at Austin.
*E-mail address:* natasa@math.utexas.edu

R. Seiringer, Institute of Science and Technology Austria (IST Austria).
*E-mail address:* robert.seiringer@ist.ac.at